On monodromy of monodromy surfaces

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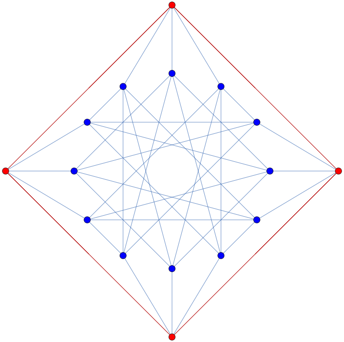
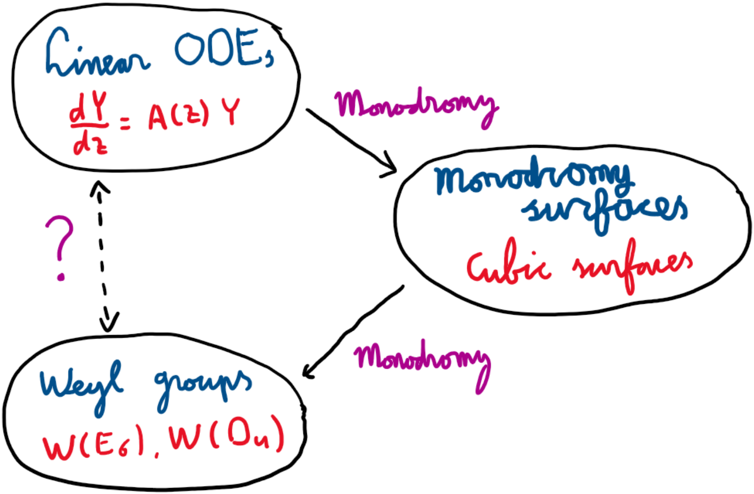


Based on joint work with Alexander Stokes (Waseda University, Tokyo)

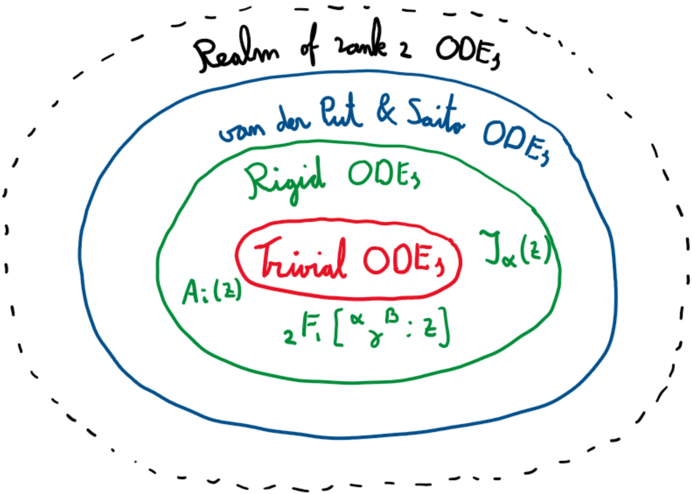
UNSW Pure Maths Seminar, Sep 2025

Overview


Linear ODEs
$\frac{dY}{dz} = A(z) Y$
Monodromy
Monodromy surfaces
Cubic surfaces
?
Monodromy
Weyl groups
$W(E_6), W(D_4)$

Van der Put & Saito families of ODEs

The **linear**, **rank 2**, ODEs on $\mathbb{CP}^1$ we focus on are **ten families** obtained in a classification by van der Put & Saito (2009).



Plan talk

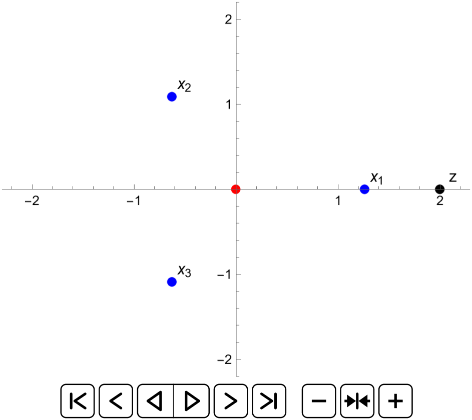
1. Monodromy of ODEs
2. Monodromy of Cubic Surfaces
3. Monodromy of Monodromy

Origins of Monodromy

Monodromy originates in Riemann's work (~1850s) on algebraic functions, complex integrals and ODEs over $\mathbb{C}$.

It describes how objects change when analytically continued along closed loops that encircle **singularities**.

Solutions of $\mathbf{x^3 - z = 0}$ are cyclically permuted as z goes around $z = 0$.



$$x_1 = \sqrt[3]{z}$$

$$x_2 = \zeta_3 \sqrt[3]{z}$$

$$x_3 = \zeta_3^{-1} \sqrt[3]{z}$$

Monodromy of logarithm

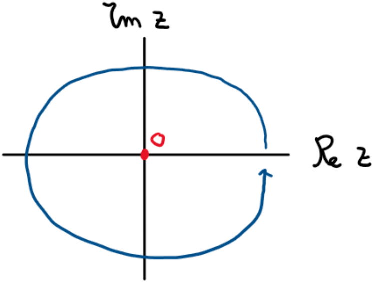
The complex logarithm,

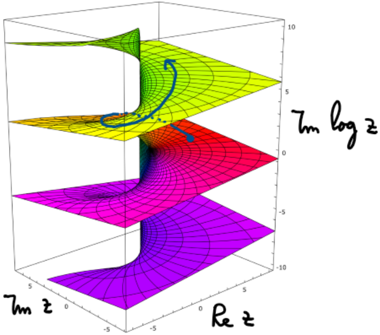
$$\log z = \int_1^z \frac{1}{x}\,\mathrm{d}x,$$

undergoes

$$\log z \mapsto \log z + 2\pi i$$

as z goes around $z = 0$ once counterclockwise.





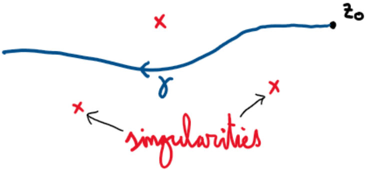
Solving ODEs along curves

Consider a linear ODE

$$\frac{dY}{dz} = A(z)Y, \qquad (z \in \mathbb{C}),$$

where $A(z)$ a matrix of rational functions over $\mathbb{C}$.

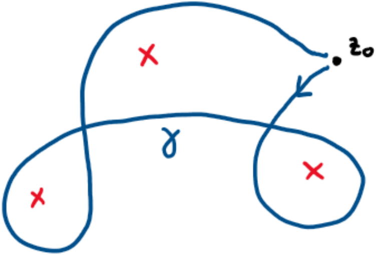
- Both A are Y are $n \times n$ matrices here.
- Given a curve γ starting at a point z_0 that avoids **singularities** of $A(z)$,



and an initial value $Y(z_0) = Y_0 \in GL_n(\mathbb{C})$, the ODE can be uniquely solved along γ.

Monodromy of an ODE

If we pick a closed loop γ,



we get the **monodromy matrix**

$$M_{\gamma} = Y(\text{start})^{-1} Y(\text{end}).$$

This defines the **monodromy representation**

$$\pi_1(\mathbb{CP}^1 - \{\text{singularities}\}, z_0) \to GL_n(\mathbb{C}), \gamma \mapsto M_{\gamma}.$$

GIT quotient

- A small wrinkle: monodromy representation **depends** on choice of Y_0!
- Changing $Y_0 \mapsto \widetilde{Y}_0 = Y_0 \cdot R$, for some $R \in GL_n(\mathbb{C})$, equates to **overall conjugation**

$$M_{\gamma} \mapsto \widetilde{M}_{\gamma} = R^{-1} M_{\gamma} R.$$

- Taking the corresponding **quotient** lands us in the **character variety**

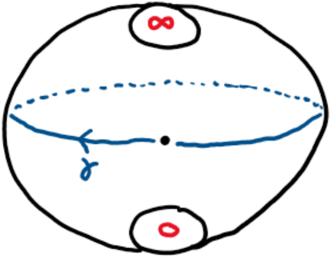
$$\mathrm{Hom}\left(\pi_1(\mathbb{CP}^1 - \{\text{singularities}\}), GL_n(\mathbb{C})\right) // GL_n(\mathbb{C}).$$

If $\mathrm{Tr}\, A(z) = 0$, then the monodromy matrix sits in $SL_n(\mathbb{C})$ due to Jacobi's formula,

$$\frac{d}{dz}|Y(z)| = \mathrm{Tr}\, A(z)\, |Y(z)| = 0.$$

Example: two singularities $\{0, \infty\}$

ODE: $\frac{dY}{dz} = \frac{A_0}{z} Y, \qquad A_0 \in \mathfrak{sl}_2(\mathbb{C}).$



Singularities: $z = \mathbf{0}$ and $z = \infty$, as change of coordinates $x = 1/z$ gives

$$\frac{dY}{dx} = \frac{dY}{dx}\frac{dz}{dx} = -\frac{A_0}{x} Y.$$

- Solution is given by

$$Y(z) = e^{\log z\, A_0} Y_0.$$

- As z follows γ around $z = 0$,

$$Y(z) = e^{\log z\, A_0} Y_0 \mapsto e^{(\log z + 2\pi i)\, A_0} Y_0 = e^{\log z\, A_0} e^{2\pi i\, A_0} Y_0$$
$$= Y(z) \left(Y_0^{-1} e^{2\pi i\, A_0} Y_0 \right).$$

- So the **monodromy matrix** is given by

$$M_\gamma = Y_0^{-1} e^{2\pi i A_0} Y_0.$$

Taking quotient w.r.t. conjugation

- Monodromy representation

$$F_1 = \langle \gamma \rangle \to SL_2(\mathbb{C}), \gamma^n \mapsto M_\gamma^n.$$

- We quotient by conjugation through the trace

$$\operatorname{Tr} M_\gamma = \operatorname{Tr} e^{2\pi i A_0} = 2\cos(2\pi\theta_0), \qquad \operatorname{Spec}(A_0) = \{+\theta_0, -\theta_0\}.$$

This is the corresponding element of the **character variety**

$$SL_2(\mathbb{C})/\!/SL_2(\mathbb{C}) \cong \mathbb{C}.$$

Lemma

For any map $r : SL_2(\mathbb{C}) \to \mathbb{C}$, that is polynomial in entries, and **invariant** *under conjugation,*

$$r(g) = R(\operatorname{Tr} g), \qquad \textit{for all } g \in SL_2(\mathbb{C}),$$

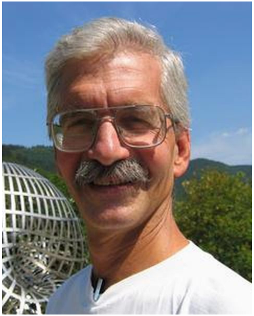
for some polynomial R over $\mathbb{C}$.

Does monodromy relate to your life?

From **Nick Katz**'s defunct **www.monodromy.com**.

Does monodromy relate to your life? It does to nearly everyone's, but surprisingly few realize it.

Do you feel that you are going around in circles and not getting anywhere? Things may not be as bad as they seem. You might be getting somewhere, but not realizing it because you aren't aware of your personal monodromy.

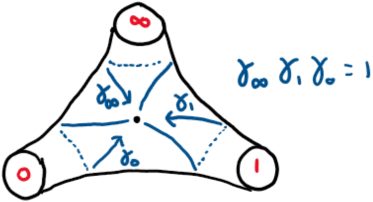


Do you think you are exactly the same person you were half your lifetime ago? If not, it is almost certainly because you are aware, at some level, of your personal monodromy. Think how much richer and more fulfilling life would be if you were completely aware of all the monodromy which surrounds you.

Three singularities $\{0, 1, \infty\}$

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right) Y,$$

$$A_0, A_1 \in \mathfrak{sl}_2(\mathbb{C}).$$



- This the **hypergeometric differential equation**, studied by Gauss, Riemann, Kummer, Klein, Poincaré, Schwarz,...
- To consider $z = \infty$, we change coordinates $x = 1/z$ and find

$$\frac{dY}{dx} = \left(\frac{A_\infty}{x} + \frac{A_1}{x-1}\right) Y, \qquad A_\infty := -(A_0 + A_1).$$

- Denote spectra

$$\mathrm{Spec}(A_j) = \{+\theta_j, -\theta_j\} \qquad (j = 0, 1, \infty).$$

- ODE can be solved in terms of **Gauss hypergeometric functions**,

$${}_2F_1\left[\begin{matrix} \frac{1}{2} \pm \theta_0 + \theta_1 + \theta_\infty, \frac{1}{2} \pm \theta_0 + \theta_1 - \theta_\infty \\ 1 \pm 2\theta_0 \end{matrix}; z\right].$$

Monodromy and rigidity

- The **monodromy** of the hypergeometric differential equation was famously computed by Riemann (1857).
- We have three monodromy matrices

$$M_0, M_1, M_\infty \in SL_2(\mathbb{C}), \qquad M_\infty M_1 M_0 = I,$$

defined up to overall conjugation.

- Generally, orbit is completely determined by the three traces,

$$\operatorname{Tr} M_0 = 2\cos(2\pi\theta_0), \quad \operatorname{Tr} M_1 = 2\cos(2\pi\theta_1), \quad \operatorname{Tr} M_\infty = 2\cos(2\pi\theta_\infty).$$

This triple forms corresponding element of **character variety**

$$SL_2(\mathbb{C})^2 /\!/ SL_2(\mathbb{C}) \cong \mathbb{C}^3.$$

Rigidity of hypergeometric differential equation

Conjugacy classes of local monodromy matrices determine monodromy representation globally up to equivalence.

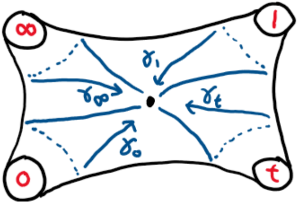
Rigid differential equations were studied extensively by Deligne and Katz.

Four singularities $\{0, t, 1, \infty\}$

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1} \right) Y,$$

$$A_0, A_t, A_1 \in \mathfrak{sl}_2(\mathbb{C}),$$

$$A_\infty := -(A_0 + A_t + A_1).$$



$$\gamma_\infty \gamma_1 \gamma_t \gamma_0 = 1$$

- This ODE has been studied well over a hundred years due to its connection with the **sixth Painlevé equation**.
- Influential works by Jimbo, Manin, Hitchin, Dubrovin, Forrester, Boalch, Lisovyy,...
- Applications in conformal field theory, random matrix theory, general relativity, quantum cohomology,...
- We again denote spectra

$$\mathrm{Spec}(A_j) = \{+\theta_j, -\theta_j\} \qquad (j = 0, t, 1, \infty).$$

Monodromy surface

- We have four monodromy matrices

$$M_0, M_t, M_1, M_\infty \in SL_2(\mathbb{C}), \qquad M_\infty M_1 M_t M_0 = I,$$

defined up to overall conjugation.

- Generally, an orbit is determined by values of the seven traces

$$\begin{array}{llll} y_0 = \mathrm{Tr}\, M_0, & y_t = \mathrm{Tr}\, M_t, & y_1 = \mathrm{Tr}\, M_1, & y_\infty = \mathrm{Tr}\, M_\infty, \\ \mathbf{x_1} = \mathrm{Tr}\, M_0 M_t, & \mathbf{x_2} = \mathrm{Tr}\, M_0 M_1, & \mathbf{x_3} = \mathrm{Tr}\, M_t M_1, & \end{array}$$

which satisfy the single constraint

$$\mathbf{x_1 x_2 x_3} + \mathbf{x_1^2} + \mathbf{x_2^2} + \mathbf{x_3^2} - w_1 \mathbf{x_1} - w_2 \mathbf{x_2} - w_3 \mathbf{x_3} + w_4 = 0, \qquad (*)$$

where

$$\begin{array}{ll} w_1 = y_0 y_\infty + y_t y_1, & w_2 = y_t y_\infty + y_0 y_1, \\ w_3 = y_1 y_\infty + y_0 y_t, & w_4 = y_0^2 + y_t^2 + y_1^2 + y_\infty^2 + y_0 y_t y_1 y_\infty - 4. \end{array}$$

Monodromy surface

- Fricke and Klein (1897), Vogt (1889), derived the character variety

  $$SL_2(\mathbb{C})^3 /\!/ SL_2(\mathbb{C}) \cong \{(\boldsymbol{x}, \mathbf{y}) \in \mathbb{C}^7 : (*)\}.$$

  A great modern reference: Goldman (2009) - *Trace coordinates...*

- From the ODE's point of view, $\mathbf{y}$ is known,

  $$y_j = 2\cos(2\pi\theta_j) \quad (j = 0, t, 1, \infty).$$

  It forms the "rigid" part of the monodromy.

- This motivates a different perspective, considering the **monodromy surface**

  $$\mathcal{M}_{w(\theta)} = \{\boldsymbol{x} \in \mathbb{C}^3 : \boldsymbol{x_1 x_2 x_3} + \boldsymbol{x_1^2} + \boldsymbol{x_2^2} + \boldsymbol{x_3^2} - w_1 \boldsymbol{x_1} - w_2 \boldsymbol{x_2} - w_3 \boldsymbol{x_3} + w_4 = 0\}.$$

  It is a four parameter family of **affine cubic surfaces**, often called the Jimbo-Fricke cubic.

Cubic surfaces

A **cubic surface** over $\mathbb{C}$ is given by the vanishing locus

$$\mathbb{V}(f) = \{X \in \mathbb{CP}^3 : f(X) = 0\}$$

of an irreducible homogeneous cubic polynomial $f \in \mathbb{C}[X_0, X_1, X_2, X_3]$.

Cayley and Salmon (1849) showed that every **smooth** cubic surface contains exactly **27 lines**.

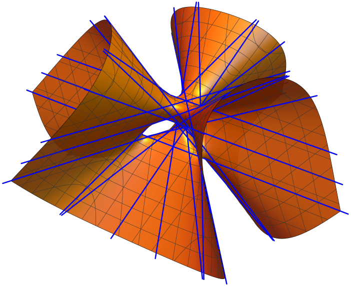
The Clebsch cubic,

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 - (X_0 + X_1 + X_2 + X_3)^3 = 0,$$

has all 27 lines defined over $\mathbb{R}$.

Example of a line is

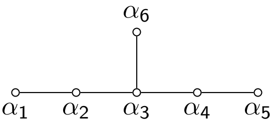
$$\{X \in \mathbb{CP}^3 : X_0 + X_1 = 0, X_2 + X_3 = 0\}.$$

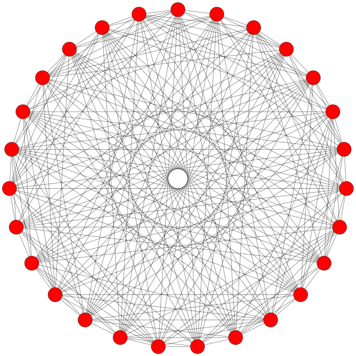


Intersection configuration of lines

Intersections among the lines in a generic smooth cubic are encoded by the **Schläfli graph** $\mathcal{G}_{\mathrm{sch}}$:

- 27 vertices are the lines,
- edges indicate disjoint lines.





Automorphism group of the Schläfli graph is the **Weyl group** $W(E_6)$.

$$W(E_6) = \left\langle r_1, \ldots, r_6 \quad \middle| \quad r_i^2 = 1, \quad \begin{array}{ll} (r_i r_j)^2 = 1 & \text{when } \underset{\alpha_i}{\circ}\ \ \underset{\alpha_j}{\circ} \\ (r_i r_j)^3 = 1 & \text{when } \underset{\alpha_i}{\circ}\!\!-\!\!\underset{\alpha_j}{\circ} \end{array} \right\rangle$$

Monodromy of smooth cubics

- A homogeneous cubic in four variables has 20 free coefficients.
- **Parameter space** of cubics: $\mathbb{P}^{19}_{\text{cubics}} \ni f, \quad C = \mathbb{V}(f) \subset \mathbb{P}^3.$
- **Incidence variety** of lines on cubics:

$$\begin{tikzcd}
\Gamma = \left\{(f,\ell) \in \mathbb{P}^{19}_{\text{cubics}} \times \mathrm{Gr}(1,\mathbb{P}^3) \mid \ell \subset \mathbb{V}(f)\right\} \arrow[d, "\pi"] \\
\mathbb{P}^{19}_{\text{cubics}}
\end{tikzcd}$$

- Denote **Locus** of **singular cubics** by $S \subset \mathbb{P}^{19}_{\text{cubics}}$. Then $\Gamma \setminus \pi^{-1}(S)$ is a covering space of $\mathbb{P}^{19}_{\text{cubics}} \setminus S$, fibres have 27 points.
- Permutations of lines as one deforms in Γ over closed loops in $\mathbb{P}^{19}_{\text{cubics}} \setminus S$ yields **monodromy group** Mon_π.

Theorem (Schläfli, Jordan, Cartan, Coble, du Val,...)

$$\mathrm{Mon}_\pi \cong \mathrm{Gal}_\pi \cong \mathrm{Aut}(\mathcal{G}_{sch}) \cong W(E_6).$$

Galois group

- Dominant morphism π induces embedding of function field

$$\pi^* : \mathbb{C}(\mathbb{P}^{19}_{\text{cubics}}) \hookrightarrow \mathbb{C}(\Gamma).$$

- Gal_π is **Galois group** of normal closure of corresponding field extension.

- Jordan (1870) first discussed Galois theory in the context of several classical problems in **enumerative geometry** and showed

$$\mathrm{Gal}_\pi \cong W(E_6).$$

- Harris (1979) revisited Jordan's work and proved that

$$\mathrm{Mon}_\pi \cong \mathrm{Gal}_\pi$$

  in each of the classical problems.

- Since Harris' paper, Galois and monodromy groups have become big in enumerative geometry, see e.g. survey Sottile-Yahl (2021).

Characterisation of monodromy surface

The affine cubic surface

$$\mathcal{M}_w = \{x \in \mathbb{C}^3 : x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - w_1 x_1 - w_2 x_2 - w_3 x_3 + w_4 = 0\},$$

has a **triangle of lines** at infinity consisting of only smooth points in its canonical projective completion $\overline{\mathcal{M}}_w \subset \mathbb{P}^3$,

$$\overline{\mathcal{M}}_w \setminus \mathcal{M}_w = \{[X_0 : X_1 : X_2 : X_3] \in \mathbb{P}^3 : X_0 = 0, X_1 X_2 X_3 = 0\},$$

where

$$[X_0 : X_1 : X_2 : X_3] = [1 : x_1 : x_2 : x_3].$$

Any affine cubic surface with this property is linearly equivalent to $\mathcal{M}_w$ for some $w \in \mathbb{C}^4$, unique up to $S_3 \ltimes K_4 \cong S_4$ action, where

- S_3 acts by permuting $\{w_1, w_2, w_3\}$.
- K_4 acts through sign flips like $(w_1, w_2, w_3) \mapsto (-w_1, -w_2, w_3)$.

$$\implies \text{Coarse moduli space}: \quad \operatorname{Spec} \mathbb{C}[w_1, w_2, w_3, w_4] // S_4.$$

Galois extension

Recall $w = w(\theta)$.

The variables

$$u_1 = e^{2\pi i(\theta_1+\theta_t)}, \quad u_2 = e^{2\pi i(\theta_0+\theta_1)}, \quad u_3 = e^{2\pi i(\theta_0+\theta_t)}, \quad u_4 = e^{2\pi i(\theta_1+\theta_\infty)},$$

define a field extension

$$\mathbb{C}(w_1, w_2, w_3, w_4) \subset \mathbb{C}(u_1, u_2, u_3, u_4)$$

that is **minimal** in allowing all lines to be written **rationally**.

$$\begin{aligned}
w_1 &= u_1 + \frac{1}{u_1} + \frac{u_2}{u_3} + \frac{u_3}{u_2} + \frac{u_2}{u_4} + \frac{u_4}{u_2} + \frac{u_1}{u_3 u_4} + \frac{u_3 u_4}{u_1},\\
w_2 &= u_2 + \frac{1}{u_2} + \frac{u_1}{u_3} + \frac{u_3}{u_1} + \frac{u_1}{u_4} + \frac{u_4}{u_1} + \frac{u_2}{u_3 u_4} + \frac{u_3 u_4}{u_2},\\
w_3 &= u_3 + \frac{1}{u_3} + u_4 + \frac{1}{u_4} + \frac{u_1}{u_2} + \frac{u_2}{u_1} + \frac{u_1 u_2}{u_3 u_4} + \frac{u_3 u_4}{u_1 u_2},\\
&\vdots
\end{aligned}$$

Lines on monodromy surface

$$\begin{aligned}
&L_k : x_1 = b_k + \frac{1}{b_k}, \qquad && b_k x_2 + x_3 = \frac{b_k w_3 - w_2}{b_k - b_k^{-1}}, \qquad && k = 1, \ldots, 8,\\
&L_k : x_2 = b_k + \frac{1}{b_k}, \qquad && b_k x_3 + x_1 = \frac{b_k w_1 - w_3}{b_k - b_k^{-1}}, \qquad && k = 9, \ldots, 16,\\
&L_k : x_3 = b_k + \frac{1}{b_k}, \qquad && b_k x_1 + x_2 = \frac{b_k w_2 - w_1}{b_k - b_k^{-1}}, \qquad && k = 17, \ldots, 24,
\end{aligned}$$

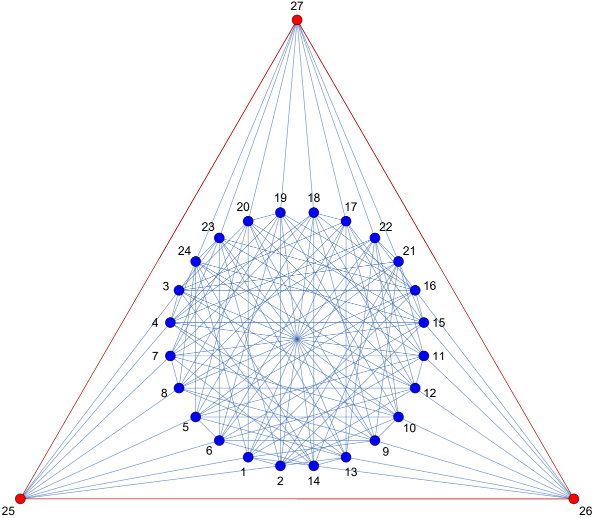
where the b_k, $k = 1, \ldots, 24$, are given by

$$\begin{array}{llllll}
b_1 = \frac{u_3}{u_2}, & b_2 = \frac{u_2}{u_3}, & b_3 = \frac{u_3 u_4}{u_1}, & b_4 = \frac{u_1}{u_3 u_4}, & b_5 = u_1, & b_6 = \frac{1}{u_1},\\
b_7 = \frac{u_2}{u_4}, & b_8 = \frac{u_4}{u_2}, & b_9 = \frac{u_1}{u_3}, & b_{10} = \frac{u_3}{u_1}, & b_{11} = \frac{u_3 u_4}{u_2}, & b_{12} = \frac{u_2}{u_3 u_4},\\
b_{13} = u_2, & b_{14} = \frac{1}{u_2}, & b_{15} = \frac{u_1}{u_4}, & b_{16} = \frac{u_4}{u_1}, & b_{17} = \frac{u_2}{u_1}, & b_{18} = \frac{u_1}{u_2},\\
b_{19} = u_4, & b_{20} = \frac{1}{u_4}, & b_{21} = u_3, & b_{22} = \frac{1}{u_3}, & b_{23} = \frac{u_1 u_2}{u_3 u_4}, & b_{24} = \frac{u_3 u_4}{u_1 u_2}.
\end{array}$$

Intersection configuration of lines

Intersection graph $\mathcal{G}_{\mathrm{aff}}$

- **blue vertices**: affine lines
- **red vertices**: lines at infinity
- **edges**: intersection points



Monodromy group of monodromy surface

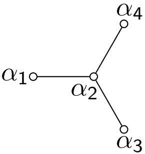
Theorem (Stokes, PR)

$$\mathrm{Mon}(\mathcal{M}) \cong \mathrm{Gal}(\mathbb{C}(u)/\mathbb{C}(w)) \cong \mathrm{Aut}(\mathcal{G}_{aff})_{(25,26,27)} \cong W(D_4).$$

The last equality was first computed in MAGMA during the MAGMA Mondays workshops at USYD. Can be computed by hand in $\mathrm{Pic}(\overline{\mathcal{M}})$.

Explicit realisation of $\mathrm{Aut}(\mathcal{G}_{\mathrm{aff}})_{(25,26,27)}$:

$$\begin{aligned} r_1 &= (3\ 7)\,(4\ 8)\,(11\ 15)\,(12\ 16)\,(19\ 23)\,(20\ 24),\\ r_2 &= (5\ 8)\,(6\ 7)\,(13\ 16)\,(14\ 15)\,(21\ 24)\,(22\ 23),\\ r_3 &= (1\ 5)\,(2\ 6)\,(9\ 14)\,(10\ 13)\,(19\ 24)\,(20\ 23),\\ r_4 &= (3\ 8)\,(4\ 7)\,(9\ 13)\,(10\ 14)\,(17\ 22)\,(18\ 21). \end{aligned}$$



Here

$$W(D_4) = \left\langle r_1, r_2, r_3, r_4 \quad \middle| \quad r_i^2 = 1, \quad \begin{array}{ll} (r_i r_j)^2 = 1 & \text{when } \ \underset{\alpha_i}{\circ}\ \ \underset{\alpha_j}{\circ} \\ (r_i r_j)^3 = 1 & \text{when } \ \underset{\alpha_i}{\circ}\!\!-\!\!\!-\!\!\underset{\alpha_j}{\circ} \end{array} \right\rangle$$

Painlevé VI

- Recall ODE
$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}\right) Y.$$
- R. Fuchs (1905) discovered that ODE can be uniquely **deformed** with respect to t such that **monodromy** is left **invariant**!
- By choosing appropriate coordinates, deformation is governed by the **sixth Painlevé equation**,

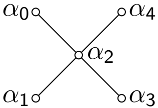
$$u_{tt} = \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\frac{u_t^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right) u_t$$
$$+ \frac{u(u-1)(u-t)}{2t^2(t-1)^2}\left((2\theta_\infty - 1)^2 - \frac{4\theta_0^2 t}{u^2} + \frac{4\theta_1^2(t-1)}{(u-1)^2} + \frac{(1-4\theta_t^2)t(t-1)}{(u-t)^2}\right).$$

- This is the most general nonlinear second order (rational) ODE that has no movable branch points.
- It is an **integrable system**, integrals are given by $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})$,
$$\frac{d\boldsymbol{x_1}}{dt} = \frac{d\boldsymbol{x_2}}{dt} = \frac{d\boldsymbol{x_3}}{dt} = 0.$$

Symmetry group of Painlevé VI

- The sixth Painlevé equation has **affine Weyl group symmetry**

$$W(D_4^{(1)}) = \langle r_0, r_1, r_2, r_3, r_4 \rangle \,,$$



  that can be further extended with Dynkin diagram automorphisms $\mathrm{Aut}(D_4^{(1)}) \cong S_4$.

- Finite $W(D_4)$ sits in there as

$$W(D_4) \subset W(D_4^{(1)}) \cong W(D_4) \ltimes T_Q,$$

  where T_Q translations on root lattice of D_4.

- The sixth Painlevé equation has an initial value space that is a **Sakai surface**. It is effectively

$$\mathcal{X}_{t,\theta} = \left\{ \frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1} \right) Y : \mathrm{Spec}(A_j) = \{+\theta_j, -\theta_j\} \right\} \Big/ \sim,$$

  where the quotient is with respect to rational equivalence.

A shadow of Weyl group under Riemann-Hilbert

Theorem (Inaba, Iwasaki and Saito (2003))

For any $g \in W(D_4^{(1)})$, the induced action on $\mathcal{M}_w$ under the Riemann-Hilbert map is trivial, i.e. the following diagram commutes.

$$\begin{tikzcd}
\mathcal{X}_{t,\theta} \arrow[r, "g"] \arrow[d, "\mathrm{RH}_{t,\theta}"] & \mathcal{X}_{t,\tilde{\theta}} \arrow[d, "\mathrm{RH}_{t,\tilde{\theta}}"] \\
\mathcal{M}_w \arrow[r, "\mathrm{id}"] & \mathcal{M}_w
\end{tikzcd}$$

However, there is a **shadow** of the finite part of the Weyl group symmetry under the Riemann-Hilbert map, it is the monodromy group of the monodromy surface,

$$\mathrm{Mon}(\mathcal{M}) \cong W(D_4).$$

Result so far

- Nine of the ten families of ODEs involve **irregular** singularities.
- For irregular ODEs, classical monodromy data is naturally extended by **Stokes data** (so that RH is injective once again)
- In each case, the monodromy surface is given as a **quotient** of a space of extended monodromy data by a Lie group.
- We require slightly finer quotients than usual, which **preserve** the original **embedding** of the space of monodromy data.
- In each case, we obtain a class of affine **Del Pezzo surfaces** characterised by a particular divisor at infinity (or a double cover of one), and its moduli space.

Result so far

The monodromy group of the monodromy surface for 6/10 of the van der Put & Saito ODEs is equal to the finite Weyl group symmetry of the corresponding Sakai surface.

We still need to work out some of the details for 4/10 of them.

Table with the data

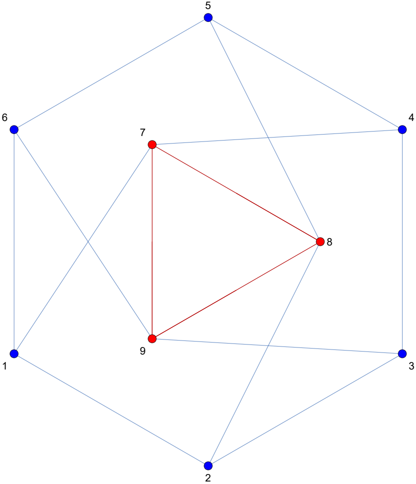
Linear ODE type	**Pain. Eq.**	**Sym. Group**	**Deg. of Del Pezzo**	**At infinity**	**monodromy of monodromy**
$(0,0,0,0)$	P_{VI}	$W(D_4^{(1)})$	3	triangle of lines	$W(D_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0,2)$	P_{IV}	$W(A_2^{(1)})$	4	rectangle of lines	$W(A_2)$
$(1,1)$	P_{III}	$W(2A_1^{(1)})$	6	conic + quartic	$W(2A_1)$
$(0,3/2)$	P_{II}	$W(A_1^{(1)})$	6	triangle of conics	$W(A_1)$
(3)	P_{II}	$W(A_1^{(1)})$	6	triangle of conics	$W(A_1)$
$(5/2)$	P_{I}	$\{1\}$	5	pentagon of lines	$\{1\}$

ODE type (3) - second Painlevé equation

$$\frac{dY}{dz} = (A_0 + z\,A_1 + z^2 A_2)\,Y.$$

$$\begin{cases} x_1 x_2 x_3 - x_1 - x_2 - x_3 + w = 0, \\ x_4 - x_2 x_3 = 0, \\ x_5 - x_1 x_3 = 0, \\ x_6 - x_1 x_2 = 0. \end{cases}$$

$$u^2 - w\,u + 1 = 0$$



$$\mathrm{Mon}(\mathcal{M}) \cong \mathrm{Gal}(\mathbb{C}(u)/\mathbb{C}(w)) \cong \mathrm{Aut}(\mathcal{G}_{\mathrm{aff}})_{(7,8,9)} \cong W(A_1) \cong \mathbb{Z}/2\mathbb{Z}.$$

ODE type $(0,2)$ - fourth Painlevé equation

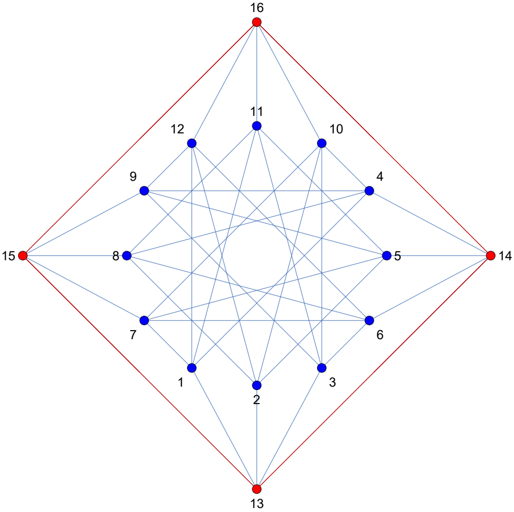
$$\frac{dY}{dz} = \left(\frac{A_{-1}}{z} + A_0 + z\,A_1 \right) Y.$$

$$\begin{cases} x_2\,x_3 + x_1 + x_4 - w_1 = 0, \\ x_1\,x_4 + x_2 + x_3 - w_2 = 0. \end{cases}$$

$$w_1 = u_1\,u_2 + \frac{1}{u_1} + \frac{1}{u_2},$$

$$w_2 = \frac{1}{u_1\,u_2} + u_1 + u_2,$$

$$u^3 - w_2\,u^2 + w_1\,u - 1 = 0.$$



$$\mathrm{Mon}(\mathcal{M}) \cong \mathrm{Gal}(\mathbb{C}(u)/\mathbb{C}(w)) \cong \mathrm{Aut}(\mathcal{G}_{\mathrm{aff}})_{(13,14,15,16)} \cong W(A_2) \cong S_3.$$

Future

1. At the moment, everything is on a case by case basis. General theory?
2. Can we classify which (embedded) affine Del Pezzo surfaces appear as monodromy surfaces?
3. Beyond dimension 2? Fano varieties?
4. What about q-difference equations? Elliptic difference equations?