On q-Painlevé VI, singular Segre surfaces and associated orthogonal polynomials

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Based on joint work with Nalini Joshi:

'*On the Riemann-Hilbert problem for a q-difference Painlevé equation*' - CMP (2021)

'*On the monodromy manifold of q-Painlevé VI and its Riemann-Hilbert problem*'

Plan talk

1. Introduction: Painlevé equations and cubic surfaces
2. *q*-Painlevé VI and a Segre surface
3. Singularities on Segre surface and OPs

Classical Painlevé equations

The classical Painlevé equations, P_{I},...,P_{VI}, are 2nd order **nonlinear** ODEs in the complex plane,

$$u_{tt} = R(u, u_t, t),$$

with R rational, without **movable** branch points.
That is, for any (local) parametrisation of the solution space,

$$u(t) = u(t; \eta),$$

the locations of the branch points of u are independent of η.

P_{I}, the simplest to write down, $u_{tt} = 6u^2 - t$.
P_{VI}, the most involved:

$$u_{tt} = \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\frac{u_t^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u_t + \frac{u(u-1)(u-t)}{2t^2(t-1)^2}\left((\theta_\infty - 1)^2 - \frac{\theta_0^2 t}{u^2} + \frac{\theta_1^2(t-1)}{(u-1)^2} + \frac{t(t-1)(1-\theta_t^2)}{(u-t)^2}\right),$$

where $\theta = (\theta_0, \theta_t, \theta_1, \theta_\infty)$ complex parameters.

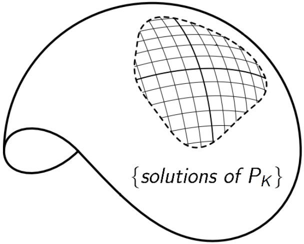
Parametrising solution spaces

How to parametrise the solution space of a Painlevé equation?

- A local method: fix a point t_0 in the complex plane and specify

$$u(t_0) = \eta_1, \quad u'(t_0) = \eta_2, \quad (\eta_1, \eta_2) \in \mathbb{C}^2.$$

  This does not cover full solution space, as e.g. $u(t)$ can have a pole at $t = t_0$. Okamoto (1979) constructed full spaces of initial conditions for the Painlevé equations.
- A global method: via **Riemann-Hilbert correspondence**.



Monodromy manifolds

- Each Painlevé equation P_K, $K = \mathrm{I}, \ldots \mathrm{VI}$, is integrable: it has an associated linear system

$$Y_z = A_K(z; u, u_t, t)Y,$$

such that, as t moves, the **monodromy data** of the linear system are preserved. [Flaschka and Newell 1980, Jimbo et al 1981]

- This yields a one-to-one correspondence

$$\textit{solutions of } P_K \leftrightarrow \textit{monodromy data.}$$

- The collection of monodromy data

$$M_K = \{\textit{monodromy data}\},$$

is called the corresponding **monodromy manifold**.

Monodromy manifolds as algebraic surfaces

Each of these monodromy manifolds M_K can be identified with an **affine cubic surface**

$$M_K \cong \{\eta \in \mathbb{C}^3 : R_K(\eta) = 0\} \qquad (R_K \text{ a cubic polynomial}).$$

Therefore, we have a (generically) one-to-one correspondence

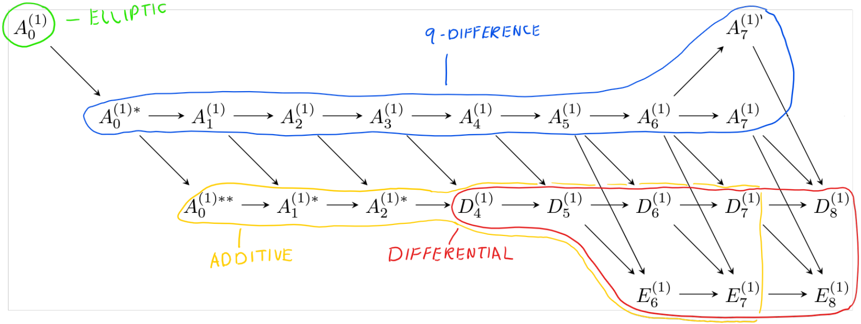
$$\{\textit{solutions of } P_K\} \to \{\eta \in \mathbb{C}^3 : R_K(\eta) = 0\}.$$

P-eqs	polynomials
P_{VI}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + \eta_3^2 + w_1\eta + w_2\eta_2 + w_3\eta_3 + w_4$
P_{V}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + R(w_{1,2,3})$
$P_{\mathrm{V}}^{\mathrm{deg}}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_1 - 1$
P_{IV}	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 + w_2(\eta_2 + \eta_3) + w_2(1 + w_1 - w_2)$
$P_{\mathrm{III}}^{D_6}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_1 - 1$
$P_{\mathrm{III}}^{D_7}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 - \eta_2$
$P_{\mathrm{III}}^{D_8}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 - \eta_2$
$P_{\mathrm{II}}^{\mathrm{JM}}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 - \eta_2 - 1$
$P_{\mathrm{II}}^{\mathrm{FN}}$	$\eta_1\eta_2\eta_3 - \eta_1 + w_2\eta_2 - \eta_3 - w_2 + 1$
P_{I}	$\eta_1\eta_2\eta_3 - \eta_1 - \eta_2 + 1$

See e.g. van der Put and Saito (2009) and Chekhov et al. (2015).

Discrete Painlevé equations

Sakai (2001) classified discrete Painlevé equations according to surface type.



- Within green: elliptic Painlevé.
- Within blue: q-difference Painlevé.
- Within yellow: additive Painlevé.
- Within red: differential Painlevé.

q-Painlevé VI

Fix $q \in \mathbb{C}$ with $0 < |q| < 1$. Then q-Painlevé VI is given by

$$q\mathrm{P}_{\mathrm{VI}}: \begin{cases} f\overline{f} &= \dfrac{(\overline{g}-\kappa_0 t)(\overline{g}-\kappa_0^{-1}t)}{(\overline{g}-\kappa_\infty)(\overline{g}-q^{-1}\kappa_\infty^{-1})}, \\ g\overline{g} &= \dfrac{(f-\kappa_t t)(f-\kappa_t^{-1}t)}{q(f-\kappa_1)(f-\kappa_1^{-1})}, \end{cases}$$

where

- $f, g : T \to \mathbb{CP}^1$, with $T \subseteq \mathbb{C}^*$ a domain invariant under multiplication by q.
- $f = f(t)$, $g = g(t)$,$\overline{f} = f(qt)$, $\overline{g} = g(qt)$, for $t \in T$.
- $\kappa = (\kappa_0, \kappa_t, \kappa_1, \kappa_\infty) \in \mathbb{C}^4$ are nonzero complex parameters.

Today, we mostly consider domains T given by a discrete q-spiral,

$$T = q^{\mathbb{Z}} t_0 = \{\ldots, q^{+2}t_0, q^{+1}t_0, t_0, q^{-1}t_0, q^{-2}t_0, \ldots\},$$

for some $t_0 \in \mathbb{C}^*$. We call (f, g) a solution of $q\mathrm{P}_{\mathrm{VI}}(\kappa, t_0)$.

Origin of q-Painlevé VI

- Around 1910, Birkhoff and his student Carmichael studied the 'Riemann problem' for q-difference Fuchsian linear systems,

$$Y(qz) = A(z)Y(z),$$
$$A(z) = A_0 + zA_1 + \ldots + z^n A_n, \quad A_0, A_n \in GL_k(\mathbb{C}),$$

  of general degree $n \geq 1$ and rank $k \geq 1$.
- The **monodromy** of such a system is encapsulated by a single connection matrix $P(z)$ relating Frobenius-type series solutions near $z = 0$ and $z = \infty$,

$$Y_\infty(z) = Y_0(z)P(z).$$

- Jimbo and Sakai (1996) showed that the case $k = n = 2$ is governed by $q\mathrm{P}_{\mathrm{VI}}$. They derived a parametrisation of such systems,

$$Y(qz) = A(z; t, f, g, \kappa)Y(z),$$

  so that P is invariant under deforming $t \mapsto qt$ iff (f, g) satisfy $q\mathrm{P}_{\mathrm{VI}}$.

Jimbo-Sakai linear system

$$Y(qz) = A(z; t, f, g, \kappa) Y(z)$$
$$A(z) = A_0 + zA_1 + z^2 A_2,$$

where

$$A_0 \sim \begin{pmatrix} \kappa_0 t & 0 \\ 0 & \kappa_0^{-1} t \end{pmatrix}, \quad A_2 = \begin{pmatrix} \kappa_\infty & 0 \\ 0 & \kappa_\infty^{-1} \end{pmatrix},$$

and

$$|A(z)| = (z - \kappa_t^{+1} t)(z - \kappa_t^{-1} t)(z - \kappa_1^{+1})(z - \kappa_1^{-1}),$$

with $t \in q^{\mathbb{Z}} t_0$.

Monodromy manifold

The Birkhoff connection matrix P can be factorised as

$$P(z) = z^{\log_q(z/qt_0) - \log_q(\kappa_0)\sigma_3} C(z) z^{\log_q(\kappa_\infty)\sigma_3},$$

where

(1) $C(z)$ is analytic and single-valued on $\mathbb{C}^*$.

(2) $C(qz) = z^{-2} \begin{pmatrix} t_0\kappa_0 & 0 \\ 0 & t_0\kappa_0^{-1} \end{pmatrix} C(z) \begin{pmatrix} \kappa_\infty^{-1} & 0 \\ 0 & \kappa_\infty \end{pmatrix}$.

(3) $|C(z)| = \text{constant} \times \theta_q(z/x_1)\,\theta_q(z/x_2)\,\theta_q(z/x_3)\,\theta_q(z/x_4)$, where $(x_1, x_2, x_3, x_4) := (\kappa_t t_0, \kappa_t^{-1} t_0, \kappa_1, \kappa_1^{-1})$.

(4) $C(z)$ is only rigidly defined up to arbitrary left and right-multiplication by diagonal matrices.

Define the **monodromy manifold** $\mathcal{M}(\kappa, t_0)$ as the space of matrices $C(z)$ satisfying (1)-(3), quotiented by arbitrary left and right-multiplication by diagonal matrices.

$$\theta_q(z) = (z;q)_\infty (q/z;q)_\infty, \quad (z;q)_\infty = \prod_{k\geq 0}(1 - q^k z).$$

Riemann-Hilbert problem

The Jimbo-Sakai linear system yields a (generically) bijective mapping

$$\{\text{solutions of } q\mathrm{P}_{\mathrm{VI}}(\kappa, t_0)\} \to \mathcal{M}(\kappa, t_0).$$

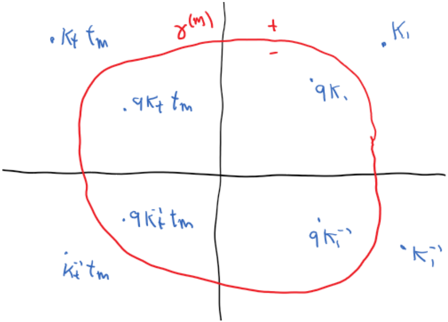
Upon fixing some monodromy $[C] \in \mathcal{M}(\kappa, t_0)$, and an $m \in \mathbb{Z}$, one can compute the value of the corresponding solution (f, g) at $t = q^m t_0$, by solving the following Riemann-Hilbert problem.

Find a 2×2 matrix valued function which satisfies:

(i) $Y^{(m)}(z)$ is analytic on $\mathbb{C} \setminus \gamma^{(m)}$.

(ii) $Y_+^{(m)}(z) = Y_-^{(m)}(z)\, z^m C(z)$.

(iii) $Y^{(m)}(z) = I + O(z^{-1})$ as $|z| \to \infty$.



q-Painlevé VI and an affine Segre surface

Theorem (Informally, Joshi and PR 2022)

For generic parameter values (κ, t_0), the monodromy manifold $\mathcal{M}(\kappa, t_0)$ of q-$P_{VI}(\kappa, t_0)$ can be identified with the affine algebraic surface

$$\mathcal{F}(\kappa, t_0) = \{\eta \in \mathbb{C}^4 : R_1(\eta) = 0 \text{ and } R_2(\eta) = 0\},$$

where R_1 and R_2 are explicit quadratic polynomials defined below.

P-eq	polynomials
q-P_{VI}	$R_1 = u_0\eta_1^2 + u_1\eta_1\eta_2 + u_2\eta_1\eta_3 + u_3\eta_1\eta_4 + u_4\eta_3\eta_4 + u_5\eta_1$ $R_2 = v_0\eta_2^2 + v_1\eta_1\eta_2 + v_2\eta_2\eta_3 + v_3\eta_2\eta_4 + v_4\eta_3\eta_4 + v_5\eta_2$

Intersections of two quadrics in $\mathbb{CP}^4$ are known as a Segre surfaces. They were introduced by Corrado Segre (1884). The surface $\mathcal{F}(\kappa, t_0)$ is an **affine Segre surface**.

Explicit formulas for coefficients

$$u_0 = \kappa_\infty^2 \theta_q \left(\kappa_t^2, \kappa_1^2, t_0 \kappa_t \kappa_1, \frac{t_0}{\kappa_t \kappa_1 \kappa_\infty^2} \right), \qquad u_1 = \theta_q \left(\kappa_t^2 \kappa_1^2, \kappa_\infty^2, \frac{t_0 \kappa_1}{\kappa_t}, \frac{t_0 \kappa_t}{\kappa_1} \right),$$

$$u_2 = \kappa_t^2 \theta_q \left(\frac{\kappa_1 \kappa_\infty}{\kappa_0 \kappa_t}, \frac{\kappa_0 \kappa_1 \kappa_\infty}{\kappa_t}, \frac{t_0}{\kappa_t \kappa_1}, t_0 \kappa_t \kappa_1 \right), \quad u_3 = -\theta_q \left(\kappa_t^2 \kappa_\infty^2, \kappa_1^2, \frac{t_0}{\kappa_t \kappa_1}, t_0 \kappa_t \kappa_1 \right),$$

$$u_4 = \theta_q \left(\kappa_t^2, \kappa_1^2, \frac{t_0}{\kappa_t \kappa_1}, t_0 \kappa_t \kappa_1 \kappa_\infty^2 \right), \qquad u_5 = \frac{\theta_q(\kappa_0 \kappa_t \kappa_1 \kappa_\infty)}{\theta_q(\kappa_0)^2} \theta_q \left(\frac{\kappa_t \kappa_1 \kappa_\infty}{\kappa_0} \right),$$

and

$$v_0 = \theta_q \left(t_0 \kappa_t \kappa_1, \frac{t_0 \kappa_\infty^2}{\kappa_t \kappa_1}, \frac{t_0 \kappa_t}{\kappa_{'}}, \frac{t_0 \kappa_1}{\kappa_t} \right), \qquad v_1 = -\frac{t_0}{\kappa_t \kappa_1} \theta_q \left(t_0^2, \kappa_t^2, \kappa_1^2, \kappa_\infty^2 \right),$$

$$v_2 = -\theta_q \left(\frac{t_0 \kappa_t}{\kappa_1}, \frac{t_0 \kappa_1 \kappa_\infty^2}{\kappa_t}, t_0 \kappa_t \kappa_1, \frac{t_0}{\kappa_t \kappa_1} \right), \quad v_3 = -\theta_q \left(\frac{t_0 \kappa_1}{\kappa_t}, \frac{t_0 \kappa_t \kappa_\infty^2}{\kappa_1}, t_0 \kappa_t \kappa_1, \frac{t_0}{\kappa_t \kappa_1} \right)$$

$$v_4 = \theta_q \left(\frac{t_0}{\kappa_t \kappa_1}, t_0 \kappa_t \kappa_1 \kappa_\infty^2, \frac{t_0 \kappa_t}{\kappa_1}, \frac{t_0 \kappa_1}{\kappa_t} \right), \qquad v_5 = \frac{t_0}{\kappa_t \kappa_1} \frac{\theta_q(t_0 \kappa_0 \kappa_\infty)}{\theta_q(\kappa_0)^2} \theta_q \left(\frac{t_0 \kappa_\infty}{\kappa_0} \right).$$

Coordinates on monodromy manifold

Take some monodromy $[C] \in \mathcal{M}(\kappa, t_0)$, then

$$|C(z)| = \text{constant} \times \theta_q(z/x_1)\theta_q(z/x_2)\theta_q(z/x_3)\theta_q(z/x_4),$$
$$(x_1, x_2, x_3, x_4) := (\kappa_t t_0, \kappa_t^{-1} t_0, \kappa_1, \kappa_1^{-1}).$$

For any 2×2 matrix R of rank 1, define

$$\pi(R) \in \mathbb{CP}^1: \quad R_1 = \pi(R) R_2, \quad R = (R_1, R_2).$$

We define **coordinates** $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)$ on monodromy manifold by

$$\rho_k = \pi(C(x_k)) \quad (1 \le k \le 4),$$

which define a point in $(\mathbb{CP}^1)^4/\mathbb{C}^*$.

A multilinear polynomial

The coordinates ρ satisfy

$$T(\rho) = 0,$$

where T is the homogeneous multi-linear polynomial

$$T(\rho) = T_{12}\rho_1\rho_2 + T_{13}\rho_1\rho_3 + T_{14}\rho_1\rho_4 + T_{23}\rho_2\rho_3 + T_{24}\rho_2\rho_4 + T_{34}\rho_3\rho_4,$$

with coefficients given by

$$\begin{aligned}
T_{12} &= \theta_q\left(\kappa_t^2, \kappa_1^2\right)\theta_q\left(\kappa_0\kappa_\infty^{-1}t_0, \kappa_0^{-1}\kappa_\infty^{-1}t_0\right)\kappa_\infty^2,\\
T_{34} &= \theta_q\left(\kappa_t^2, \kappa_1^2\right)\theta_q\left(\kappa_0\kappa_\infty t_0, \kappa_0^{-1}\kappa_\infty t_0\right),\\
T_{13} &= -\theta_q\left(\kappa_t\kappa_1^{-1}t_0, \kappa_t^{-1}\kappa_1 t_0\right)\theta_q\left(\kappa_t\kappa_1\kappa_0^{-1}\kappa_\infty^{-1}, \kappa_0\kappa_t\kappa_1\kappa_\infty^{-1}\right)\kappa_\infty^2,\\
T_{24} &= -\theta_q\left(\kappa_t\kappa_1^{-1}t_0, \kappa_t^{-1}\kappa_1 t_0\right)\theta_q\left(\kappa_0\kappa_t\kappa_1\kappa_\infty, \kappa_t\kappa_1\kappa_\infty\kappa_0^{-1}\right),\\
T_{23} &= \theta_q\left(\kappa_t\kappa_1 t_0, \kappa_t^{-1}\kappa_1^{-1}t_0\right)\theta_q\left(\kappa_t\kappa_\infty\kappa_0^{-1}\kappa_1^{-1}, \kappa_0\kappa_t\kappa_\infty\kappa_1^{-1}\right)\kappa_1^2,\\
T_{14} &= \theta_q\left(\kappa_t\kappa_1 t_0, \kappa_t^{-1}\kappa_1^{-1}t_0\right)\theta_q\left(\kappa_1\kappa_\infty\kappa_0^{-1}\kappa_t^{-1}, \kappa_0\kappa_1\kappa_\infty\kappa_t^{-1}\right)\kappa_t^2.
\end{aligned}$$

η-coordinates

The ρ-coordinates on monodromy manifold satisfy $T(\rho) = 0$ and

$$T|_{\kappa_0=1}(\rho) \neq 0.$$

The coordinates

$$\eta_1 = \frac{\rho_1\rho_2}{T|_{\kappa_0=1}(\rho)}, \quad \eta_2 = \frac{\rho_1\rho_3}{T|_{\kappa_0=1}(\rho)}, \quad \eta_3 = \frac{\rho_1\rho_4}{T|_{\kappa_0=1}(\rho)}, \quad \eta_4 = \frac{\rho_2\rho_3}{T|_{\kappa_0=1}(\rho)},$$

yields a (generically) bijective mapping

$$\mathcal{M}(\kappa, t_0) \to \mathcal{F}(\kappa, t_0),$$

from the monodromy mapping onto the affine Segre surface $\mathcal{F}$.
So we have (generically) one-to-one correspondences

$$\{\text{solutions of } q\text{-}P_{\mathrm{VI}}(\kappa, t_0)\} \to \mathcal{M}(\kappa, t_0) \to \mathcal{F}(\kappa, t_0).$$

Parametrisation of solution space

The general solution of q-$P_{\mathrm{VI}}(\kappa, t_0)$ can be parametrised as

$$f(t) = f(t; \kappa, t_0, \boldsymbol{\eta}),$$
$$g(t) = g(t; \kappa, t_0, \boldsymbol{\eta}),$$

where

- time t varies in $q^{\mathbb{Z}} t_0$,
- coordinates $\boldsymbol{\eta}$ vary in $\mathcal{F}(\kappa, t_0)$,

for parameters $(\kappa, t_0) \in (\mathbb{C}^*)^5$ away from some (explicit) hypersurfaces in $\mathbb{C}^5$.

Upon fixing a point $\boldsymbol{\eta} \in \mathcal{F}(\kappa, t_0)$ and $t \in q^{\mathbb{Z}} t_0$, the value of (f, g) at t can be computed by solving associated Riemann-Hilbert problem with $C(z) = C(z; \boldsymbol{\eta})$.

Meaning of generic parameter values

We call (κ, t_0) **generic** when the

non-resonance conditions

$$\kappa_0^2, \kappa_t^2, \kappa_1^2, \kappa_\infty^2 \notin q^{\mathbb{Z}}, \qquad (\kappa_t \kappa_1)^{\pm 1}, (\kappa_t/\kappa_1)^{\pm 1} \notin t_0 q^{\mathbb{Z}},$$

and

non-splitting conditions

$$\kappa_0^{\epsilon_0} \kappa_t^{\epsilon_t} \kappa_1^{\epsilon_1} \kappa_\infty^{\epsilon_\infty} \notin q^{\mathbb{Z}}, \quad \kappa_0^{\epsilon_0} \kappa_\infty^{\epsilon_\infty} \notin t_0 q^{\mathbb{Z}}, \quad \epsilon_j \in \{\pm 1\}, j = 0, t, 1, \infty,$$

are satisfied.
The non-resonance conditions are essential for our construction.

When the non-splitting conditions are violated, the correspondence

$$\{\text{solutions of } q\text{-}P_{\mathrm{VI}}(\kappa, t_0)\} \to \mathcal{F}(\kappa, t_0),$$

ceases to be one-to-one and $\mathcal{F}(\kappa, t_0)$ has singularities.

A singularity and family of solutions

Consider $\kappa_\infty = q^n \kappa_0 \kappa_t^{-1} \kappa_1^{-1}$, $n \geq 0$.
The Segre surface has an (ordinary double point) singularity at $\eta = 0$.
A whole family of solutions

$$f_n(t) = f_n(t;\nu), \quad g_n(t) = g_n(t;\nu), \quad \nu \in \mathbb{C}^*,$$

is mapped onto this singularity via correspondence

$$\{\text{solutions of } q\text{-}P_{\mathrm{VI}}(\kappa, t_0)\} \to \mathcal{F}(\kappa, t_0).$$

Their monodromy is parametrised by

$$C_n(z;\nu) = \begin{pmatrix} \theta_q\left(\frac{z}{\kappa_t t_0}, \frac{z}{\kappa_1}\right) z^{-n} & \theta_q\left(\frac{z}{\nu t_0}, \frac{z\nu\kappa_t\kappa_1}{\kappa_0^2}\right) z^n \\ 0 & \theta_q\left(\frac{z\kappa_t}{t_0}, z\kappa_1\right) z^n \end{pmatrix},$$

and in particular **triangular**.

Simplification of Riemann-Hilbert Problem

The corresponding Riemann-Hilbert problem (RHP) can be recast into Fokas-Its-Kitaev form for orthogonal polynomials, with complex weight function

$$w(z,t) = \frac{\theta_q\left(\frac{z}{\nu t}, \frac{z\nu}{\kappa_0\kappa_\infty}\right)}{\left(\frac{z}{\kappa_t t}, \frac{z}{\kappa_1}; q\right)_\infty \left(\frac{qt}{\kappa_t z}, \frac{q}{\kappa_1 z}; q\right)_\infty}.$$

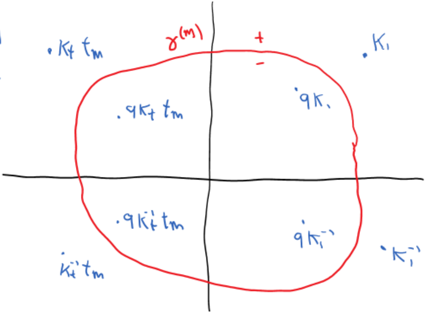
For $m \in \mathbb{Z}$, $n \geq 0$, find a matrix-valued function $Y(z) = Y^{(m,n)}(z)$ which satisfies:

(i) $Y(z)$ is analytic in $\mathbb{C} \setminus \gamma^{(m)}$.

(ii) $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w(z;t_m) \\ 0 & 1 \end{pmatrix}$

(iii) $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$

as $|z| \to \infty$.



Explicit solvability RHP

The RHP can be solved explicitly in terms of a family of orthogonal polynomials with respect to the complex inner product

$$\begin{aligned}\langle h_1, h_2\rangle = &+ \alpha_1(t_m,\nu)\int_0^{qt_m/\kappa_t} h_1(z)h_2(z)W(z,t_m)d_qz\\ &+ \alpha_2(t_m,\nu)\int_0^{q/\kappa_1} h_1(z)h_2(z)W(z,t_m)d_qz,\end{aligned}$$

where

$$W(z,t) = z^{\sigma}\frac{\left(\frac{\kappa_t z}{t}, \kappa_1 z; q\right)_\infty}{\left(\frac{z}{\kappa_t t}, \frac{z}{\kappa_1}; q\right)_\infty}, \quad \sigma := 2\log_q(\kappa_0),$$

$$\alpha_1(t,\nu) = \frac{(\kappa_t/t)^{\sigma}}{(1-q)(q;q)_\infty^2}\frac{\theta_q\left(\frac{1}{\kappa_t\nu}, \frac{\kappa_1 t\nu}{\kappa_0^2}\right)}{\theta_q\left(\frac{\kappa_1 t}{\kappa_t}\right)},$$

$$\alpha_2(t,\nu) = \frac{(\kappa_1)^{\sigma}}{(1-q)(q;q)_\infty^2}\frac{\theta_q\left(\frac{1}{\kappa_1\nu t}, \frac{\kappa_t\nu}{\kappa_0^2}\right)}{\theta_q\left(\frac{\kappa_t}{\kappa_1 t}\right)}.$$

These OPs were studied by Ormerod et al. (2012) for a special value of ν.

Explicit formula for moments

The kth moment $\mu_k(t;\nu) := \langle z^k, z^k \rangle$ is given by

$$\mu_k(t;\nu) = S_1 + S_2,$$

where

$$S_1 = \frac{\kappa_0^2 \theta_q(q\kappa_t\nu)}{(q;q)_\infty (q/\kappa_t^2;q)_\infty} \frac{\left(q^{1+k}\frac{q\kappa_0^2}{\kappa_t^2};q\right)_\infty}{(q^{1+k}\kappa_0^2;q)_\infty} \left(\frac{qt}{\kappa_t}\right)^{k+1} \frac{\theta_q\left(\frac{\kappa_1\nu t}{\kappa_0^2}\right)}{\theta_q\left(\frac{\kappa_1 t}{\kappa_t}\right)}$$
$$\times\ {}_2\phi_1\left[\begin{matrix}\kappa_1^2, q^{1+k}\kappa_0^2 \\ q^{2+k}\frac{\kappa_0^2}{\kappa_t^2}\end{matrix}; q, \frac{qt}{\kappa_t\kappa_1}\right],$$

$$S_2 = \frac{\kappa_0^2 \theta_q\left(\frac{\kappa_t\nu}{\kappa_0^2}\right)}{\nu\kappa_t(q;q)_\infty (q/\kappa_1^2;q)_\infty} \frac{\left(q^{1+k}\frac{q\kappa_0^2}{\kappa_1^2};q\right)_\infty}{(q^{1+k}\kappa_0^2;q)_\infty} \left(\frac{q}{\kappa_1}\right)^{k+1} \frac{\theta_q(\kappa_1\nu t)}{\theta_q\left(\frac{\kappa_1 t}{\kappa_t}\right)}$$
$$\times\ {}_2\phi_1\left[\begin{matrix}\kappa_t^2, q^{1+k}\kappa_0^2 \\ q^{2+k}\frac{\kappa_0^2}{\kappa_1^2}\end{matrix}; q, \frac{q}{\kappa_t\kappa_1 t}\right].$$

Corresponding solution of q-P_{VI}

The solution can be written explicitly as

$$f_n(t) = \frac{\kappa_\infty^2 - 1}{q\kappa_\infty^2 - 1}\frac{\Gamma_n(t)}{\Delta_n(t)} - \frac{q^2\kappa_\infty - 1}{q\kappa_\infty^2 - 1}\frac{\Gamma_{n+1}(t)}{\Delta_{n+1}(t)} + L(t),$$

$$g_n(t) = \kappa_\infty \frac{\nu\Delta_n(t/q)\Delta_{n+1}(t) - \kappa_t\Delta_n(t)\Delta_{n+1}(t/q)}{\nu\Delta_n(t/q)\Delta_{n+1}(t) - \kappa_t\Delta_n(t)\Delta_{n+1}(t/q)q\kappa_\infty^2},$$

$$L(t) = \kappa_t t + \kappa_1 + \frac{\kappa_t(\kappa_1^2 - 1) + t\kappa_1(\kappa_t^2 - 1)}{\kappa_t\kappa_1(q\kappa_\infty^2 - 1)}.$$

Here Δ_n is the nth Hankel determinant of moments

$$\Delta_n(t) := \det\left[(\mu_{i+j}(t))_{0\le i,j\le n-1}\right],$$

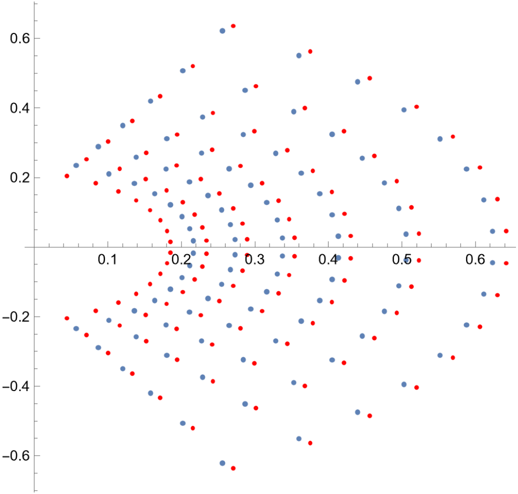
and

$$\Gamma_n(t) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \\ \mu_{n-2} & \mu_{n-1} & \cdots & \mu_{2n-4} & \mu_{2n-2} \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}.$$

These formulas were first derived by Ormerod, Witte and Forrester (2012).

Reduction to rational case

Setting $\nu = \frac{\kappa_0^2}{\kappa_t}$ and $\kappa_1 = q^{-\frac{1}{2}r}$, $r \in \mathbb{N}$, yields rational solutions (f, g).



Distribution of poles of f in t-plane for $n = 6$, $r = 16$ and particular choices of remaining parameters.