On the Asymptotic distribution of Roots of the Generalised Hermite Polynomials

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Based on two joint works with Davide Masoero (University of Lisbon):

Roots of generalised Hermite polynomials when both parameters are large, ArXiv, 2019;
Poles of Painlevé IV Rationals and their Distribution, SIGMA, 2018.

Overview

Painlevé Functions

- According to Wikipedia, **special functions** *are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications.*
- According to Iwasaki et al. - From Gauss to Painlevé: a modern theory of special functions (1991), **Painlevé functions** *are the* **nonlinear** **special functions** *of the 21st century.*
- **Painlevé functions** have applications in many fields involving some form of **nonlinearity**:
  - general relativity
  - nonlinear wave equations
  - nonlinear optics
  - random matrix theory
  - quantum mechanics
  - statistical mechanics (conformal field theory)

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Classical Special Functions

- Most often, a classical **special function** is an **analytic** function which solves a **linear** second order ODE and admits an **integral representation**.
- Example, hypergeometric function ${}_2F_1(a, b, c; z)$

  ODE:

  $$z(z-1)\omega_{zz} + [c - (a+b+1)z]\omega_z - ab\omega = 0,$$

  integral representation for $z \in (0,1)$:

  $${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt.$$

- What about using **nonlinear ODEs** to define nonlinear special functions?

Linearity $\implies$ singularities fixed

As to **linear** differential equations, one can read of the equation itself directly where its solutions might be singular. i.e. its **singularities** are **fixed**. Example, hypergeometric equation

$$z(z-1)\omega_{zz} + [c-(a+b+1)z]\omega_z - ab\omega = 0$$
$$\Downarrow$$
$$\{z \in \mathbb{P}^1(\mathbb{C}) : \omega(z) \text{ singular}\} \subseteq \{0, 1, \infty\}.$$

On the contrary, **nonlinear** differential equations generically have **movable singularities**. Example:

$$\omega_z = \frac{\omega - \omega^3}{z(z+1)}$$

has general solution

$$\omega(z) = c\left(\frac{1+z}{1+c^2 z}\right)^{\frac{1}{2}}, \quad c = \omega(0).$$

- $z = -1$ is a **fixed branch point**,
- $z = -c^{-2}$ is a **movable branch point**.

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nonlinear special functions

- An ODE is said to have the **Painlevé property** if it does **not** have any **movable branch points nor movable essential singularities**.
- solutions → functions.
- Around 1900, Painlevé, Gambier and R. Fuchs classified all **second order** ODEs of the form

$$\omega_{zz} = R(\omega, \omega_z, z),$$

  with $R(\omega, \omega_z, z)$ rational in ω, ω_z and entire in z, satisfying the **Painlevé property**.
- Result: all such ODEs can be transformed into one of six canonical equations, the six Painlevé equations, or reduced to linear or first order equations.

The Six Painlevé Equations

Result of classification:

$$
\begin{aligned}
P_{\mathrm{I}}: \quad & \omega_{zz} = 6\omega^2 + z, \\
P_{\mathrm{II}}: \quad & \omega_{zz} = 2\omega^3 + z\omega + \alpha, \\
P_{\mathrm{III}}: \quad & \omega_{zz} = \frac{1}{\omega}\omega_z^2 - \frac{1}{z}\omega_z + \frac{1}{z}(\alpha\omega^2 + \beta) + \gamma\omega^3 + \frac{\delta}{\omega}, \\
P_{\mathrm{IV}}: \quad & \omega_{zz} = \frac{1}{2\omega}\omega_z^2 + \frac{3}{2}\omega^3 + 4z\omega^2 + 2(z^2 - \alpha)\omega + \frac{\beta}{\omega}, \\
P_{\mathrm{V}}: \quad & \omega_{zz} = \left(\frac{1}{2\omega} + \frac{1}{\omega - 1}\right)\omega_z^2 - \frac{1}{z}\omega_z + \frac{(\omega-1)^2}{z^2}\left(\alpha\omega + \frac{\beta}{\omega}\right) + \\
& \qquad \frac{\gamma}{z}\omega + \delta\frac{\omega(\omega+1)}{\omega - 1}, \\
P_{\mathrm{VI}}: \quad & \omega_{zz} = \frac{1}{2}\left(\frac{1}{\omega} + \frac{1}{\omega - 1} + \frac{1}{\omega - z}\right)\omega_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{\omega - z}\right)\omega_z + \\
& \qquad \frac{\omega(\omega-1)(\omega-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{\omega^2} + \gamma\frac{z-1}{(\omega-1)^2} + \delta\frac{z(z-1)}{(\omega-z)^2}\right).
\end{aligned}
$$

Here $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are parameters.

Painlevé Functions

- Most often, a classical **special function** is an **analytic** function which solves a **linear** second order ODE and admits an **integral representation**.
- A **Painlevé function** is a **meromorphic function** which solves a **Painlevé equation**.
- As it turns out, each Painlevé function also has a ‘nonlinear’ integral representation through an associated **Riemann-Hilbert problem**.

Introduction to Problem

Movable Poles

- Even though Painlevé equations do not have movable branch points or essential singularities, they do have **movable poles**.
- The problem of computing the distribution of poles of Painlevé functions is a long-standing open problem.
- In applications the locations of poles are often of special interest.
- Only for a limited number of Painlevé functions strong results on pole distribution have been obtained:
  - Painlevé I: the tritronquée solution, Costin et al (2014), Masoero (2010-2014).
  - Painlevé II: rational solutions, Buckingham and Miller (2014,2015), Bertola and Bothner (2015).
  - **Painlevé IV: rational solutions (Hermite)**, Buckingham (2018), PR and Masoero (2018,2019)

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Movable Poles, Pictorial examples

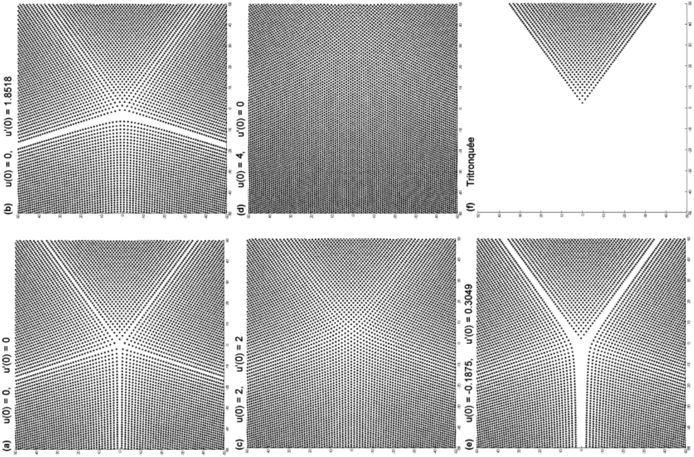


Figure: Poles solutions $P_{\mathrm{I}} : \omega_{zz} = 6\omega^2 + z$, source: Fornberg et al. (2011)

Study of poles unveil deep mathematical structures.

Movable poles of PIV

This talk is on poles of **rational solutions** to the fourth Painlevé equation, given by

$$P_{\mathrm{IV}}: \quad \omega_{zz} = \frac{1}{2\omega}\omega_z^2 + \tfrac{3}{2}\omega^3 + 4z\omega^2 + 2(z^2 + 1 - 2\theta_\infty)\omega - \frac{8\theta_0^2}{\omega},$$

where $\theta_0, \theta_\infty \in \mathbb{C}$ are complex parameters.

Movable poles:
For any $\epsilon \in \{\pm 1\}$, $a \in \mathbb{C}$, $b \in \mathbb{C}$, there exists a unique solution $\omega(z)$ with

$$\omega(z) = \frac{\epsilon}{z - a} - a + u(z - a) + b(z - a)^2 + \mathcal{O}(z - a)^3, \quad (z \to a),$$

where $u = \frac{1}{3}\epsilon(a^2 - 2 + 4\theta_\infty) - \frac{4}{3}$.

Hermite Rationals

For $m, n \in \mathbb{N}$,

$$\omega^{(\mathrm{I})}_{m,n} = \frac{H'_{m+1,n}}{H_{m+1,n}} - \frac{H'_{m,n}}{H_{m,n}}, \quad \theta_0 = \tfrac{1}{2}n, \quad \theta_\infty = m + \tfrac{1}{2}n + 1,$$

defines a rational solution of P_{IV}.
Here $H_{m,n}(z)$ are the **generalised Hermite polynomials**,

$$H_{m,n}(z) = \begin{vmatrix} h_m(z) & h_{m+1}(z) & \dots & h_{m+n-1}(z) \\ h^{(1)}_m(z) & h^{(1)}_{m+1}(z) & \dots & h^{(1)}_{m+n-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ h^{(n-1)}_m(z) & h^{(n-1)}_{m+1}(z) & \dots & h^{(n-1)}_{m+n-1}(z) \end{vmatrix},$$

where $h^{(l)}_k(z)$ denotes the l-th derivative of the k-th Hermite polynomial

$$h_k(z) = (-1)^k e^{z^2} \frac{\partial^k}{\partial z^k}\left[e^{-z^2}\right].$$

Note: poles with $+1$ and -1 residue coincide with roots of different generalised Hermite polynomials!

Generalised Hermite Polynomials

Degree:

$$\deg(H_{m,n}) = m \times n$$

Symmetry:

$$H_{n,m}(z) = i^{mn} H_{m,n}(-iz)$$

Examples:

$$\begin{aligned}
H_{m,1}(z) &= h_m(z) \quad (m \in \mathbb{N}),\\
H_{2,2}(z) &= z^4 + 12\\
H_{3,2}(z) &= z^6 - 6z^4 + 36z^2 + 72\\
H_{3,3}(z) &= z^9 + 72z^5 - 2160z\\
H_{4,2}(z) &= z^8 - 16z^6 + 120z^4 + 720\\
H_{4,3}(z) &= z^{12} - 12z^{10} + 180z^8 - 480z^6 - 3600z^4 - 43200z^2 + 43200\\
H_{4,4}(z) &= z^{16} + 240z^{12} - 7200z^8 + 2016000z^4 + 6048000
\end{aligned}$$

Roots of Generalised Hermite polynomials

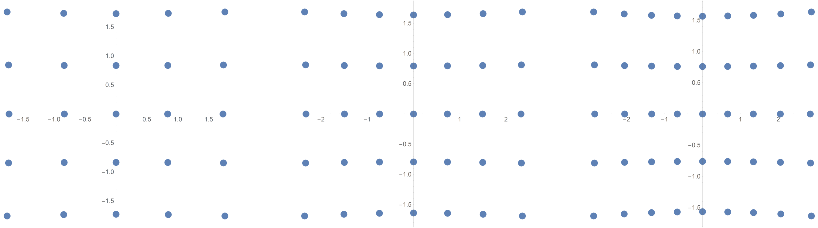


Figure: Roots of $H_{m,n}$, with $n = 5$ and $m = 5, 7, 9$

Problem (Clarkson,2003)

Explain the pictures!

Roots of Generalised Hermite polynomials

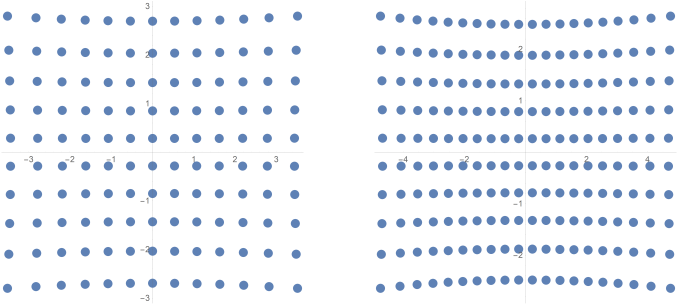


Figure: Rescaled roots of $H_{m,n}$, with $n = 10$ and $m = 13, 20$

Problem (Clarkson,2003)

Explain the pictures!

The Elliptic Region

Setting

$$E = 2m + n, \quad n = E\nu,$$

and keeping $\nu > 0$ fixed, so ratio

$$\frac{m}{n} = \frac{1-\nu}{2\nu} \quad \text{fixed},$$

the roots of

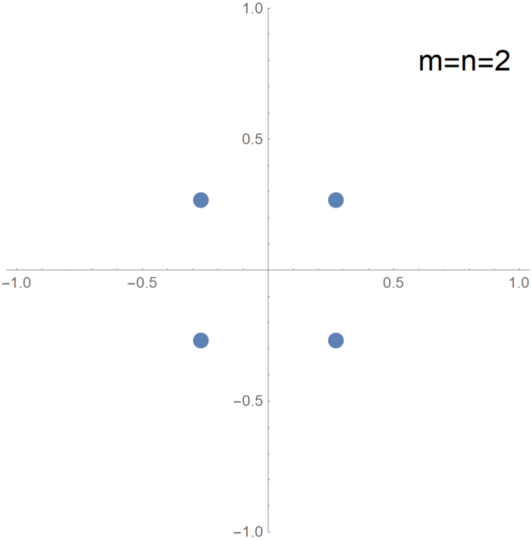
$$H_{m,n}(E^{\frac{1}{2}} z)$$

seem to condensate on compact region $K = K(\nu) \subseteq \mathbb{C}$ as $E \to \infty$.

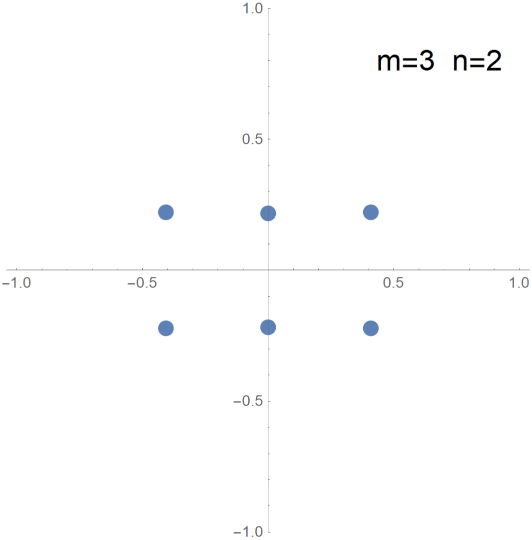
Problem

Determine the 'Elliptic region' $K = K(\nu) \subseteq \mathbb{C}$ and prove that the roots indeed densely fill this region as $E \to \infty$.

The Elliptic Region, $\nu = \frac{1}{3}$, $\frac{m}{n} = 1$



The Elliptic Region, $\nu = \frac{1}{4}$, $\frac{m}{n} = \frac{3}{2}$



An Application to Orthogonal Polynomials (Assche, 2016)

Consider the complex weight on the real line

$$w(x; z, m) = x^m e^{-x^2} e^{2izx}, \quad x \in (-\infty, +\infty),$$

with parameters $z \in \mathbb{C}$ and $m \in \mathbb{N}$.
The n-th Hankel determinant of moments

$$D_n = D_n(z, m) = \det \left(\int_{-\infty}^{+\infty} x^{j+k} w(x) dx \right)_{j,k=0}^{n-1}$$

equals

$$D_n(z, m) = c_{m,n} e^{-nz^2} H_{m,n}(z) \quad (c_{m,n} \in \mathbb{C}^*).$$

Let $(P_n(x))_{n \in \mathbb{N}}$ be the monic orthogonal polynomials w.r.t. weight $w(x; z, m)$, which exist for generic $z \in \mathbb{C}$.

Important observation:

$$P_n(x) = P_n(x; z, m) \text{ exists} \iff H_{m,n}(z) \neq 0.$$

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Method of Attack

Overview

Step 1: Apply isomonodromic deformation method.
Result: Roots $z = a$ of generalised Hermite polynomials $H_{m,n}(z)$ are inextricably linked to certain biconfluent Heun equations

$$\psi''(\lambda) = V(\lambda)\psi(\lambda)$$

$$V(\lambda) = \lambda^2 + 2a\lambda + a^2 - (2m+n) - \frac{b}{\lambda} + \frac{n^2-1}{4\lambda^2}.$$

Step 2: Analyse these biconfluent Heun equations via a complex WKB approach in the $E \to \infty$ limit.
Result: As $E \to \infty$, our original problem becomes asymptotically equivalent to a certain **model problem**.
Step 3: Solve the model problem.

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Isomonodromic Deformation Method

In beginning of eighties, expanding on work by R. Fuchs (1907), Jimbo, Miwa and Ueno showed that each **Painlevé equation** governs **isomonodromic deformation** within a specific class of linear systems.

- Classical special functions have (linear) integral representation.
- Painlevé functions have (nonlinear) integral representations through **Riemann-Hilbert problems**.
- Each Painlevé function $\omega(z)$ has an associated Riemann-Hilbert problem $\mathcal{RH}(z)$.
- At **movable pole** $z = z_0$ either solution of $\mathcal{RH}(z)$ does not exist or is degenerate. This yields correspondence between movable poles and certain **(confluent) Heun equations**.

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Classification Roots

Theorem (D. Masoero and PR, 2018)

For $m, n \in \mathbb{N}$, the point $a \in \mathbb{C}$ is a root of $H_{m,n}$ if and only if there exists a $b \in \mathbb{C}$ such that the biconfluent Heun equation

$$\psi''(\lambda) = (\lambda^2 + 2a\lambda + a^2 - (2m+n) - \frac{b}{\lambda} + \frac{n^2-1}{4\lambda^2})\psi(\lambda), \qquad (1)$$

satisfies the following two properties:

1. **Apparent Singularity Condition.** *The monodromy around Fuchsian singularity $\lambda = 0$ is scalar. In a formula,*

$$\psi(e^{2\pi i}\lambda) = (-1)^{n+1}\psi(\lambda)\,, \quad \forall\psi \text{ solution of } (1).$$

2. **Quantisation Condition.** *There exists a non-zero solution of (1) which solves the following boundary value problem*

$$\lim_{\lambda\to+\infty}\psi(\lambda) = \lim_{\lambda\to0^+}\psi(\lambda) = 0\,.$$

Rescaling

Setting

$$E = 2m + n, \quad \alpha = E^{-\frac{1}{2}} a, \quad \beta = E^{-\frac{3}{2}} b, \quad \nu = \frac{n}{E},$$

we have:

$z = \alpha$ is a root of $H_{m,n}(E^{\frac{1}{2}} z)$ if and only if $\exists \beta$ such that

$$\psi''(\lambda) = \left(E^2 V(\lambda; \alpha, \beta, \nu) - \frac{1}{4\lambda^2} \right) \psi(\lambda),$$

$$V(\lambda; \alpha, \beta, \nu) = \lambda^2 + 2\alpha\lambda + \alpha^2 - 1 - \beta\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2},$$

satisfies **apparent singularity** and **quantisation condition**.

Next step: complex WKB approach as $E \to \infty$.

Complex WKB Approach

As $E \to \infty$ solutions of biconfluent Heun equation are well-approximated by WKB functions

$$\psi = V^{-\frac{1}{4}} e^{\pm E \int^{\lambda} \sqrt{V(\mu)} d\mu},$$

$$V = \lambda^2 + 2\alpha\lambda + \alpha^2 - 1 - \beta\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2}.$$

This yields, as $E \to \infty$, that the **apparent singularity** and **quantisation condition** are asymptotically equivalent to a set of conditions,

- one geometric,
- two analytic,

on the potential $V = V(\lambda; \alpha, \beta, \nu)$.

Stokes Geometry

Consider potential

$$V(\lambda; \alpha, \beta, \nu) = \lambda^2 + 2\alpha\lambda + \alpha^2 - 1 - \beta\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2}.$$

- **Stokes lines** are level sets $\Re \int_{\lambda^*}^{\lambda} \sqrt{V(\lambda)}d\lambda = 0$ in $\mathbb{P}^1$, where λ^* any zero of $V(\lambda)$.
- **Stokes complex** $\mathcal{C} = \mathcal{C}(\alpha, \beta) \subseteq \mathbb{P}^1$ of $V(\lambda)$ is union of all its Stokes lines and zeros.

Geometric Condition on Potential

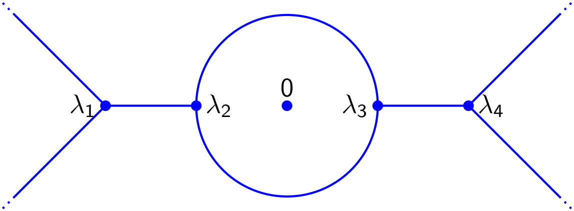

Figure: Stokes complex $\mathcal{C}(\alpha,\beta)$ with $(\alpha,\beta)=(0,0)$, where $\lambda_{1,2,3,4}$ are the zeros of $V=\lambda^2-1+\frac{\nu^2}{4}\lambda^{-2}$.

Geometric Condition on potential $V(\lambda;\alpha,\beta,\nu)$

The Stokes complex $\mathcal{C}(\alpha,\beta)$ of $V(\lambda)$ is homeomorphic to the Stokes complex $\mathcal{C}(0,0)$.

A Pair of Cycles

Consider elliptic curve

$$p^2 = \lambda^4 + 2\alpha\lambda^3 + (\alpha^2 - 1)\lambda^2 - \beta\lambda + \frac{\nu^2}{4}.$$

Assume $V(\lambda;\alpha,\beta,\nu)$ satisfies the **geometric condition**, then we can rigidly define two cycles $\boldsymbol{\gamma_{1,2}}$ on elliptic curve as in figure.

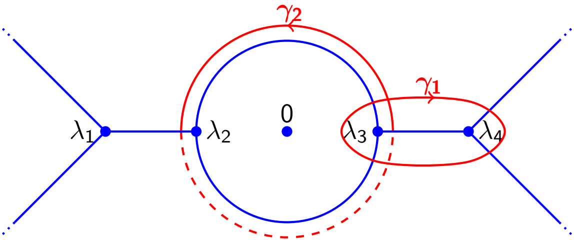

Figure: Cycles $\boldsymbol{\gamma_{1,2}}$ on elliptic curve where $\boldsymbol{\gamma_1}$ lies in sheet $p \sim +\frac{\nu}{2}$ as $\lambda \to 0$.

Two Complete Elliptic Integrals

Let $\omega := \frac{p}{\lambda} d\lambda$ be pull-back of $\sqrt{V} d\lambda$ on elliptic curve

$$p^2 = \lambda^4 + 2\alpha\lambda^3 + (\alpha^2 - 1)\lambda^2 - \beta\lambda + \frac{\nu^2}{4}.$$

Assume $V(\lambda; \alpha, \beta, \nu)$ satisfies the **geometric condition**, then we define the complete elliptic integrals

$$s_1 = \int_{\gamma_1} \omega + \frac{i\pi(1-\nu)}{2},$$
$$s_2 = \int_{\gamma_2} \omega.$$

WKB Estimate

On the cut plane $\mathbb{C} \setminus \mathbb{R}_-$, let $\psi_0, \psi_{+\infty}$ be solutions of

$$\psi''(\lambda) = \left(E^2 V(\lambda; \alpha, \beta, \nu) - \frac{1}{4\lambda^2} \right) \psi(\lambda).$$

uniquely determined up to multiplicative factors as the solutions satisfying

$$\lim_{\lambda \to 0^+} \psi_0(\lambda) = 0, \qquad \lim_{\lambda \to +\infty} \psi_{+\infty}(\lambda) = 0.$$

Then

$$\textbf{quantisation condition} \iff Wr[\psi_0, \psi_{+\infty}] = 0.$$

WKB estimate

- Let $D \subseteq \mathbb{C}^2$ be compact with $\mathcal{C}(\alpha, \beta) \simeq \mathcal{C}(0,0)$, $\forall (\alpha, \beta) \in D$.
- Then, after a suitable normalisation of $\psi_0, \psi_{+\infty}$, there exist $C_0, E_0 > 0$ such that, for all $E \geq E_0$,

$$\left| (Wr[\psi_0, \psi_{+\infty}] + 1) e^{E \cdot s_1 - i\pi m} + 1 \right| \leq \frac{C_0}{E} \qquad \forall (\alpha, \beta) \in D.$$

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$$\left|(Wr[\psi_0, \psi_{+\infty}] + 1)e^{E \cdot s_1 - i\pi m} + 1\right| \leq \frac{C_0}{E} \qquad \forall(\alpha, \beta) \in D.$$

WKB Results

quantisation condition: $Wr[\psi_0, \psi_{+\infty}] \equiv 0$ is asymptotically equivalent to

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Similarly **apparent singularity condition** is asymptotically equivalent to

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Stokes geometry+residue theorem implies $\mathfrak{R}s_{1,2} = 0$ and

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The Model Problem

Model Problem

Given $m, n \in \mathbb{N}$, $E := 2m + n$, $n = E\nu$, for $(j, k) \in I_m \times I_n$, construct a potential $V(\lambda; \alpha, \beta, \nu)$ such that

- Stokes complex $\mathcal{C}(\alpha, \beta)$ homeomorphic to $\mathcal{C}(0, 0)$,
- the following **analytic conditions** are satisfied

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Conclusion of WKB Analysis (heuristically speaking)

As $E \to \infty$, the solutions (α, β) of the **apparent singularity** and **quantisation condition** are well-approximated by $(\widetilde{\alpha}, \widetilde{\beta})$ such that potential $V(\lambda; \alpha, \beta, \nu)$ solves the **model problem**.

Note: $\#(I_m \times I_n) = m \times n = \deg H_{m,n}$.

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Solution to Model Problem

- Let $R \subseteq \mathbb{C}^2$ be set of (α, β) such that potential $V(\lambda; \alpha, \beta, \nu)$ satisfies **geometric condition** and set $K = \overline{R}$.
- Define $\mathcal{S} : R \to \mathbb{R}^2, (\alpha, \beta) \to (\Im s_1(\alpha, \beta), \Im s_2(\alpha, \beta))$.

Theorem (D. Masoero and PR (2019))

R is a regular smooth 2-dimensional real submanifold of $\mathbb{C}^2$.
$\mathcal{S}$ maps R C^∞-diffeomorphically onto the interior Q° of the quadrilateral

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Furthermore, it extends uniquely to a homeomorphism $\mathcal{S} : K \to Q$.

- **Corollary:** For every $(j,k) \in I_m \times I_n$, the model problem has a unique solution

$$(\widetilde{\alpha}_{j,k}^{(E)}, \widetilde{\beta}_{j,k}^{(E)}) = \mathcal{S}^{-1}\left[\frac{\pi j}{E}, \frac{\pi k}{E}\right].$$

- The $\widetilde{\alpha}_{j,k}^{(E)}$ are WKB approximations of roots of $H_{m,n}(E^{\frac{1}{2}} z)$.
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Results

Elliptic Region

Definition (Elliptic Region)

Let $K_a = K_a(\nu)$ be projection of $K = \overline{R}$ onto α-plane. We call K_a the **elliptic region**.

By definition, the WKB approximations $\widetilde{\alpha}_{j,k}^{(E)}$ lie in K_a for $(j,k) \in I_m \times I_n$.

Theorem (Elliptic Region, part 1)

As $E \to \infty$, roots of $H_{m,n}(E^{\frac{1}{2}} z)$ **densely fill** *up elliptic region K_a.*

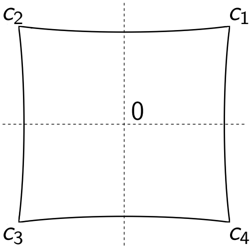
Elliptic Region, Corners

Theorem (Elliptic Region, part 2)

The elliptic region K_a is a compact quadrilateral domain whose boundary is a Jordan curve composed of four analytic pieces (edges), meeting at four corners $c_{1,2,3,4}$, as in figure.
The corner c_k is the unique solution of

$$\alpha^8 - 6(3\nu^2 + 1)\alpha^4 + 8(1 - 9\nu^2)\alpha^2 - 3(9\nu^4 + 6\nu^2 + 1) = 0$$

in k-th quadrant of complex α-plane. (Remaining four roots are purely real or imaginary)



Elliptic Region, Boundary Parametrisation

Theorem (Elliptic Region, part 3)

Let $x = x(\alpha)$ and $y = y(\alpha)$ be the unique algebraic functions which solve

$$3x^4 + 4\alpha x^3 + (\alpha^2 - 1)x^2 - \frac{\nu^2}{4} = 0, \qquad x(\alpha) \sim \frac{\nu}{2}\alpha^{-1} \quad (\alpha \to \infty),$$
$$y^2 = \alpha^2 + 6x\alpha + 6x^3 - 1, \qquad y(\alpha) \sim \alpha \quad (\alpha \to \infty),$$

on the α plane cut along diagonals $[c_1, c_3]$ and $[c_2, c_4]$. Then

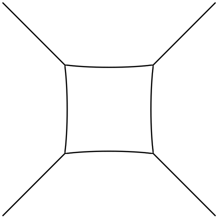
$$\psi(\alpha) = \tfrac{1}{2}\Re\left[\alpha y + \tfrac{1}{2}(1-\nu)\log(p_1) - \log(p_2) + \nu\log(x^{-2}p_3)\right],$$
$$p_1 = 1 - 2x\alpha - 2x^2, \quad p_2 = 2x + \alpha + y, \quad p_3 = x(\alpha^2 + 5x\alpha + 4x^2 - 1) + \frac{1}{2}\nu y,$$

is a **univalued harmonic function** *on the same cut plane.*

Elliptic Region, Boundary Parametrisation

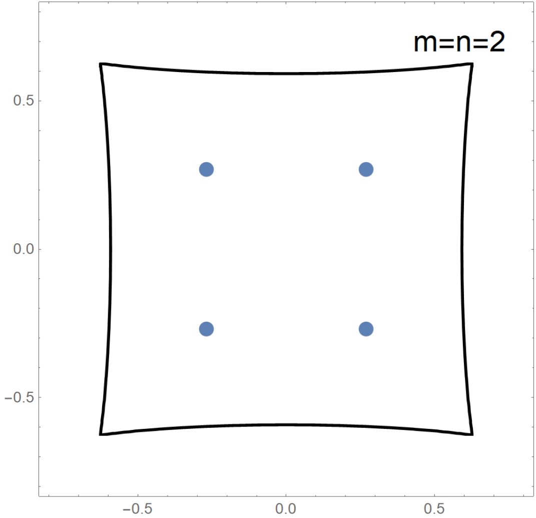
Theorem (Elliptic Region, part 4)

The level set $\{\psi(\alpha) = 0\}$ *consists of* **boundary elliptic region** ∂K_a *plus four additional lines which emanate from corners and go to infinity, see figure.*

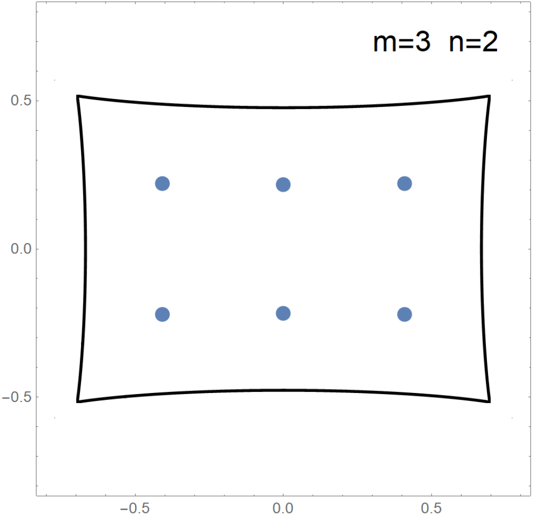


Buckingham (2018) obtained different parametrisation elliptic region via Riemann-Hilbert approach to aforementioned orthogonal polynomials and proved:
asymptotically there are **no roots outside elliptic region**.

The Elliptic Region, $\nu = \frac{1}{3}$, $\frac{m}{n} = 1$



The Elliptic Region, $\nu = \frac{1}{4}$, $\frac{m}{n} = \frac{3}{2}$



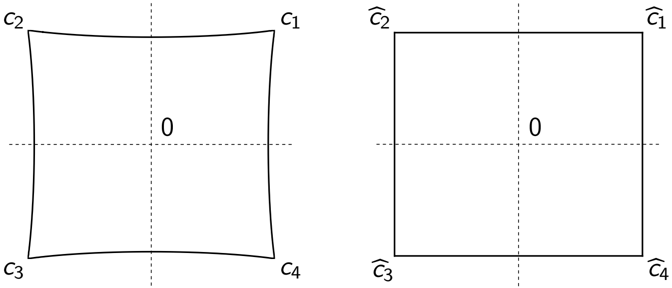
Mapping $\mathcal{S}_a : K_a \to Q$

Recall $K = \overline{R}$ and K_a is projection of K onto α-plane.

- The projection $\Pi : K \to K_a$ is a homeomorphism;
- Recall $\mathcal{S} = (\Im s_1, \Im s_2) : K \to Q$ is homeomorphism, where

$$Q := \left[-\tfrac{1}{2}(1-\nu)\pi, +\tfrac{1}{2}(1-\nu)\pi\right] \times [-\nu\pi, +\nu\pi];$$

- $\mathcal{S}_a = \mathcal{S} \circ \Pi^{-1} : K_a \to Q$ is homeomorphism and C^∞ diffeomorphism when restricted to interior of domain and co-domain.



Asymptotic Distribution (Heuristically)

Recall exact solutions $(\widetilde{\alpha}_{j,k}, \widetilde{\beta}_{j,k}) = \mathcal{S}^{-1}(\frac{\pi j}{E}, \frac{\pi k}{E})$ of model problem.

$$\widetilde{\alpha}_{j,k} = \mathcal{S}_a^{-1}(\frac{\pi j}{E}, \frac{\pi k}{E}), \quad (j,k) \in I_m \times I_n.$$

So WKB predictions $\widetilde{\alpha}_{j,k}$ are precisely the **vertices** of deformed quadrilateral **lattice** consisting of m 'vertical' and n 'horizontal' lines:

$$\mathcal{S}_a^{-1}[l_v^{(j)}], \qquad l_v^{(j)} = \{(x,y) \in Q : x = \frac{\pi j}{E}\} \qquad (j \in I_m),$$

$$\mathcal{S}_a^{-1}[l_h^{(k)}], \qquad l_h^{(k)} = \{(x,y) \in Q : y = \frac{\pi k}{E}\} \qquad (k \in I_n).$$

Asymptotic Distribution of Bulk, heuristically

In the large E limit, the bulk of the roots organise themselves within elliptic region K_a along the **vertices** of deformed **quadrilateral lattice** above.

Asymptotic Distribution (Rigorously)

Theorem (Asymptotic Distribution of Bulk)

For any $0 < \sigma < 1$, there exists $R_\sigma > 0$ such that, for E large enough:

- *Within each disc with center $E^{\frac{1}{2}}\widetilde{\alpha}_{j,k}$ and radius $E^{-\frac{3}{2}}R_\sigma$, $(j,k) \in I_m^\sigma \times I_n^\sigma$, there* **exists precisely one root** *of $H_{m,n}(z)$.*
- *These are* **all roots** *in ϵ-neighbourhood of $\mathcal{K}^\sigma$ with radius $E^{-\frac{3}{2}}R_\sigma$,*

$$\mathcal{K}^\sigma := E^{\frac{1}{2}}\mathcal{S}_a^{-1}\left(Q^\sigma\right),$$

$$Q^\sigma := \left[-\frac{\pi\lfloor\sigma(m-1)\rfloor}{E}, \frac{\pi\lfloor\sigma(m-1)\rfloor}{E}\right] \times \left[-\frac{\pi\lfloor\sigma(n-1)\rfloor}{E}, \frac{\pi\lfloor\sigma(n-1)\rfloor}{E}\right].$$

Asymptotic Distribution

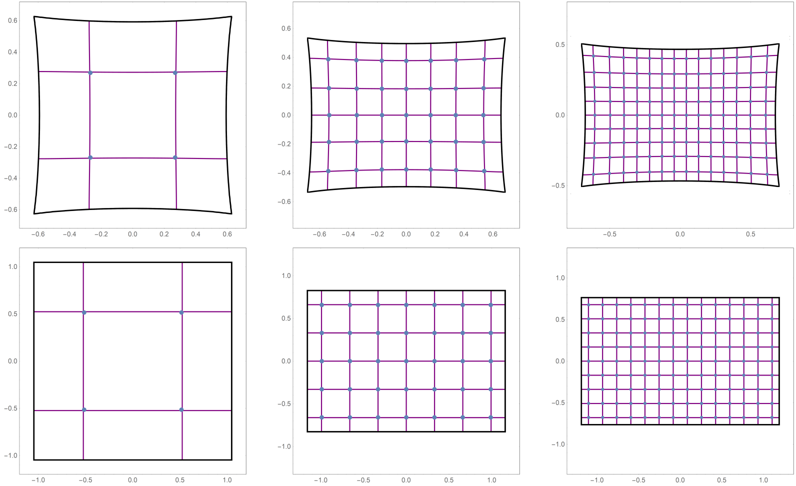


Figure: Top: asymptotic prediction in elliptic region K_a, bottom: asymptotic prediction in Q, left to right $(m, n) = (2, 2)$, $(7, 5)$, $(14, 9)$.

Asymptotic Distribution

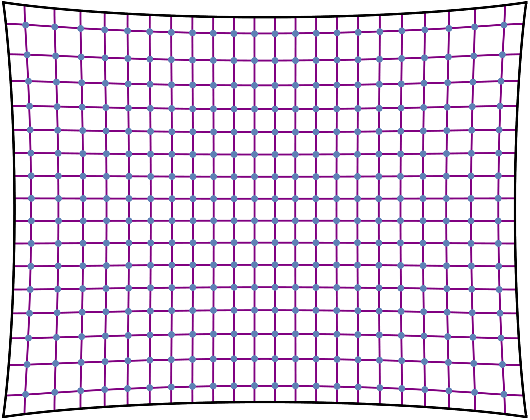


Figure: Asymptotic prediction are vertices of purple lattice, true location roots $H_{m,n}(z)$ in blue, with $(m,n) = (22,16)$.

Future

Okamoto Rationals

For $m, n \in \mathbb{Z}$,

$$\widetilde{\omega}_{m,n} = -\frac{2}{3}z + \frac{Q_{m+1,n}}{Q_{m+1,n}} - \frac{Q_{m,n}}{Q_{m,n}}, \quad \theta_0 = -\frac{1}{6} + \frac{1}{2}n, \quad \theta_\infty = \frac{1}{2}(2m + n + 1),$$

define rational solutions of P_{IV}, where $Q_{m,n}(z)$ are **generalised Okamoto polynomials** recursively defined by

$$Q_{m+1,n}Q_{m-1,n} = \frac{9}{2}\left(Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right) + (2z^2 + 3(2m + n - 1))Q^2_{m,n},$$
$$Q_{m,n+1}Q_{m,n-1} = \frac{9}{2}\left(Q_{m,n}Q''_{m,n} + \left(Q'_{m,n}\right)^2\right) + (2z^2 + 3(1 - m - 2n))Q^2_{m,n},$$

with $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2}z$.
Note: poles with $+1$ and -1 residue coincide with roots of different generalised Okamoto polynomials!

Roots of Generalised Okamoto polynomials

Problem

Explain the picture!

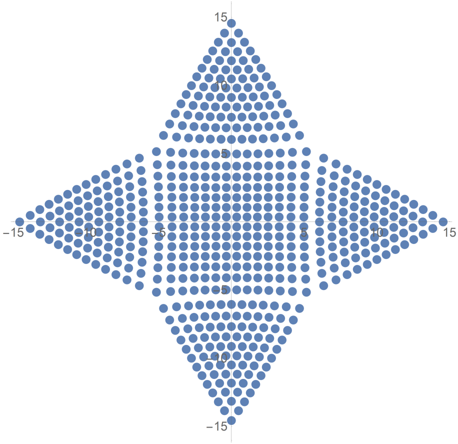
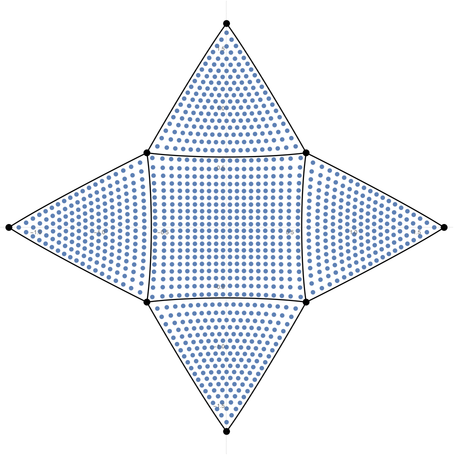


Figure: Roots of $Q_{14,14}$

Preliminary Result



Thanks for your attention!