A Riemann-Hilbert approach to q-difference Painlevé VI

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Integrable Systems and Random Matrix Theory Seminar

References

- Joshi, PR - CMP ('23)
  On the monodromy manifold of q-Painlevé VI and its RHP
- PR - arXiv:2305.17912
  On q-Painlevé VI and the geometry of Segre surfaces
- Joshi, Mazzocco, PR - (to appear)
  Segre surfaces and geometry of the Painlevé equations

Plan talk

1. Intro to classical Painlevé equations
    - Painlevé VI
    - Monodromy manifolds
2. q-difference Painlevé VI
    - Jimbo-Sakai linear problem
    - Monodromy manifold
    - Asymptotics
3. Outlook

Painlevé equations: hunt for new special functions

- Many of the classical special functions (Whittaker/Bessel/Airy functions, Hermite polynomials etc), satisfy a second order linear ODE,

$$y''(z) = a(z)y'(z) + b(z)y(z).$$

- **Linearity $\implies$ singularities are fixed:** Singularities of solutions are a subset of singularities of coefficients of ODE.
- On the other hand, considering nonlinear ODEs,

$$u_{tt} = R(u, u_t, t), \qquad R \text{ rational},$$

  solutions generally have movable branch points, i.e. their locations vary per solution and cannot be read off the ODE itself.
- Painlevé, Gambier and Picard (~ 1900), set out to classify all nonlinear second order ODEs that share the nice property with linear ODEs that the locations of singularities of solutions are fixed.

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The Painlevé Equations

Result of classification:

$$P_{\mathrm{I}}: \quad u_{tt} = 6u^2 + t,$$

$$P_{\mathrm{II}}: \quad u_{tt} = 2u^3 + tu + \alpha,$$

$$P_{\mathrm{III}}: \quad u_{tt} = \frac{1}{u}u_t^2 - \frac{1}{t}u_t + \frac{1}{t}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u},$$

$$P_{\mathrm{IV}}: \quad u_{tt} = \frac{1}{2u}u_t^2 + \frac{3}{2}u^3 + 4tu^2 + 2(t^2 - \alpha)u + \frac{\beta}{u},$$

$$P_{\mathrm{V}}: \quad u_{tt} = \left(\frac{1}{2u} + \frac{1}{u-1}\right)u_t^2 - \frac{1}{t}u_t + \frac{(u-1)^2}{t^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma}{t}u + \delta\frac{u(u+1)}{u-1},$$

$$P_{\mathrm{VI}}: \quad u_{tt} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)u_t^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u_t + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right).$$

Here $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are parameters.

The sixth Painlevé equation

- Painlevé, Gambier and Picard missed P_{VI} in their classification.
- R. Fuchs discovered P_{VI} (1905),

$$u_{tt} = \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\frac{u_t^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u_t$$
$$+\frac{u(u-1)(u-t)}{2t^2(t-1)^2}\left((2\theta_\infty - 1)^2 - \frac{4\theta_0^2 t}{u^2} + \frac{4\theta_1^2(t-1)}{(u-1)^2} + \frac{(1-4\theta_t^2)t(t-1)}{(u-t)^2}\right).$$

- P_{VI} sits at the top of the Painlevé hierarchy, with four complex parameters $\theta := (\theta_0, \theta_t, \theta_1, \theta_\infty)$, from which the others can be obtained through degeneration:

$$\begin{array}{ccccc} P_{VI} & \longrightarrow & P_V & \longrightarrow & P_{III} & & \\ & & \downarrow & & \downarrow & & \\ & & P_{IV} & \longrightarrow & P_{II} & \longrightarrow & P_I \end{array}$$

R. Fuchs' discovery of Painlevé VI

R. Fuchs (1905) was interested in constructing a **linear** second order ODE with four regular singular points, placed at $z = 0, 1, \infty, t$, whose **monodromy** is independent of t.

Earlier work by Poincaré (1883) shows that this requires an additional apparent singularity, say at $z = u$,

$$y''(z) = -\left(\frac{1}{z} + \frac{1}{z-t} + \frac{1}{z-1} - \frac{1}{z-u}\right) y'(z) + V\, y(z),$$

$$V = \frac{\theta_0^2}{z^2} + \frac{\theta_t^2}{(z-t)^2} + \frac{\theta_1^2}{(z-1)^2} + \frac{A}{z} + \frac{B}{z-t} + \frac{C}{z-1} + \frac{p}{z-u}.$$

R. Fuchs showed that isomonodromy is equivalent to $u = u(t)$ satisfying the **sixth Painlevé equation** and

$$p = \frac{(1-t)u_t}{2u} + \frac{1-u_t}{2(u-t)} + \frac{t\ u_t}{2(u-1)}.$$

Coefficients of Fuchs' ODE

Coefficients A, B, C in Fuchs' ODE are determined by

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & t & 1 \\ \frac{1}{u} & \frac{1}{u-t} & \frac{1}{u-1} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -p \\ -pu + \theta_\infty(\theta_\infty - 1) - \theta_0^2 - \theta_t^2 - \theta_1^2 \\ p^2 - p\left(\frac{1}{u} + \frac{1}{u-t} + \frac{1}{u-1}\right) - \frac{\theta_0^2}{u^2} - \frac{\theta_t^2}{(u-t)^2} - \frac{\theta_0^2}{(u-1)^2} \end{bmatrix}$$

1st line $\Longrightarrow$ $z = \infty$ is regular.

2nd line $\Longrightarrow$ exponents at $z = \infty$ are $\theta_\infty, 1 - \theta_\infty$.

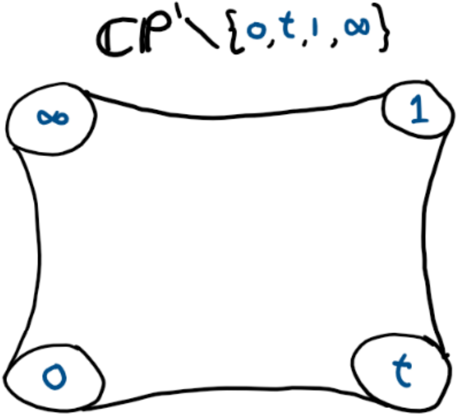
3th line $\Longrightarrow$ $z = u$ is apparent.

Local exponents are encoded in Riemann scheme

0	t	1	∞	u
$+\theta_0$	$+\theta_t$	$+\theta_1$	$+\theta_\infty$	0
$-\theta_0$	$-\theta_t$	$-\theta_1$	$1 - \theta_\infty$	2

Monodromy of Fuchs' ODE

Ley $Y = Y(z)$ be a row-vector of two linearly independent solutions around $z = z_0$.



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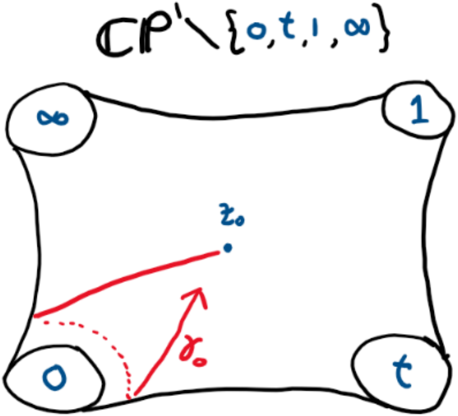

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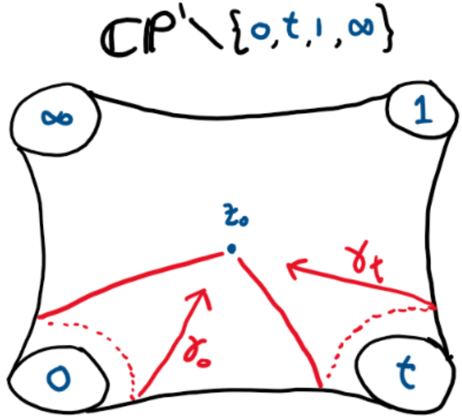

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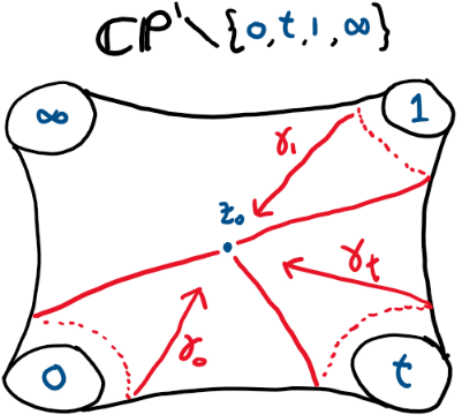

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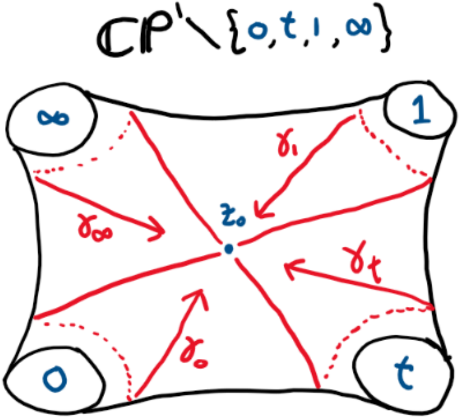

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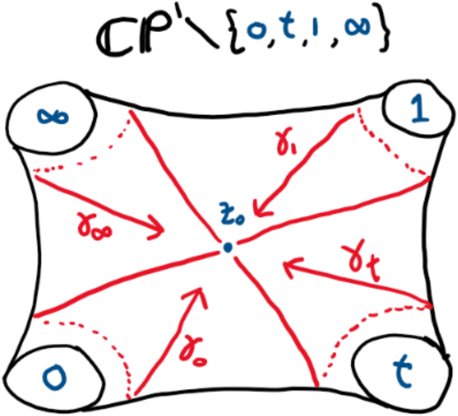
$$Y_{\gamma_j} = Y \cdot M_j,$$
$$j = 0, t, 1, \infty.$$



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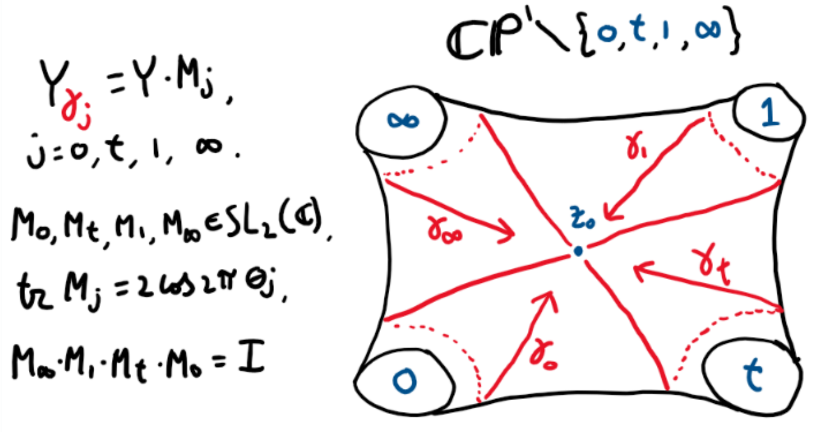

Space of monodromy data or **monodromy manifold**:

$$\mathcal{M} = \{M_{0,t,1,\infty} \in SL_2(\mathbb{C}) : M_\infty M_1 M_t M_0 = I, \operatorname{Tr} M_j = 2\cos 2\pi\theta_j\} // SL_2(\mathbb{C}).$$

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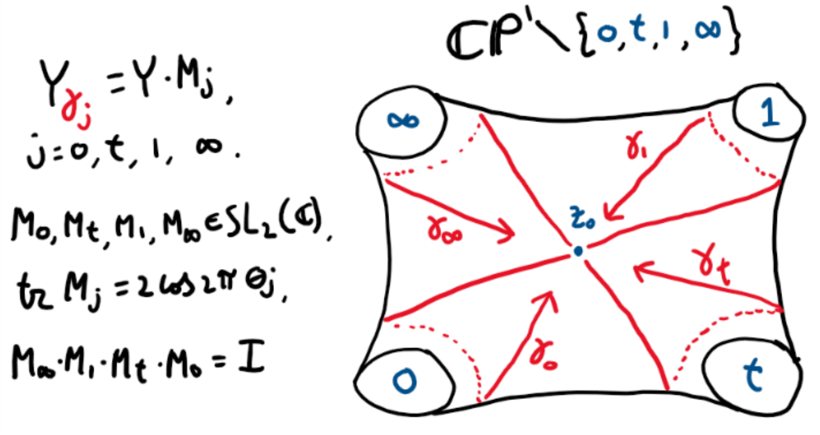

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Integrals of motion

Painlevé VI is integrable: the **trace coordinates**

$$\eta_1 = \operatorname{Tr} M_0 M_t, \qquad \eta_2 = \operatorname{Tr} M_0 M_1, \qquad \eta_3 = \operatorname{Tr} M_t M_1,$$

form a complete set of integrals of motion [Jimbo, 1982].

Fricke and Klein (1897), also Vogt (1889), showed that the trace coordinates are related by the **cubic equation**

$$R := \eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + \eta_3^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + w_4 = 0,$$

with coefficients

$$w_1 = -(r_1 r_4 + r_2 r_3), \qquad w_2 = -(r_2 r_4 + r_1 r_3),$$
$$w_3 = -(r_3 r_4 + r_1 r_2), \qquad w_4 = r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_1 r_2 r_3 r_4 - 4,$$

where $r_j = 2\cos(2\pi\theta_j)$, for $j = 0, t, 1, \infty$.

The **monodromy manifold** $\mathcal{M}$ of $\mathrm{P_{VI}}$ is isomorphic to the **affine cubic surface** $\{\eta \in \mathbb{C}^3 : R(\eta) = 0\}$.

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Monodromy manifolds

For each Painlevé equation P_K, the monodromy manifolds M_K can be identified with an **affine cubic surface**

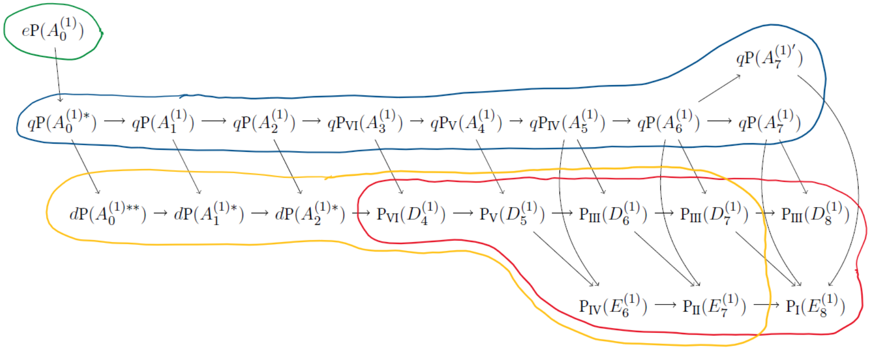
$$\mathcal{M}_K \cong \{\eta \in \mathbb{C}^3 : R_K(\eta) = 0\} \qquad (R_K \text{ a cubic polynomial}).$$

P-eqs	polynomials
P_{VI}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + \eta_3^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + w_4$
P_{V}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + R(w_{1,2,3})$
$P_{\mathrm{V}}^{\mathrm{deg}}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_1 - 1$
P_{IV}	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 + w_2(\eta_2 + \eta_3) + w_2(1 + w_1 - w_2)$
$P_{\mathrm{III}}^{D_6}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_1 - 1$
$P_{\mathrm{III}}^{D_7}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 - \eta_2$
$P_{\mathrm{III}}^{D_8}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 - \eta_2$
$P_{\mathrm{II}}^{\mathrm{JM}}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 - \eta_2 - 1$
$P_{\mathrm{II}}^{\mathrm{FN}}$	$\eta_1\eta_2\eta_3 - \eta_1 + w_2\eta_2 - \eta_3 - w_2 + 1$
P_{I}	$\eta_1\eta_2\eta_3 - \eta_1 - \eta_2 + 1$

See e.g. van der Put and Saito (2009) and Chekhov et al. (2015).

Discrete Painlevé equations

Sakai (2001) classified all Painlevé equations, differential and discrete, in terms of their initial value spaces.



- In green the elliptic Painlevé equation.
- In blue the q-difference Painlevé equations.
- In yellow the additive Painlevé equations.
- In red the classical Painlevé equations.

q-Painlevé VI

Fix $0 < q < 1$, then q-Painlevé VI is given by

$$qP_{\mathrm{VI}}: \begin{cases} f\overline{f} = \dfrac{(\overline{g} - q^{+\theta_0} t)(\overline{g} - q^{-\theta_0} t)}{(\overline{g} - q^{\theta_\infty - 1})(\overline{g} - q^{-\theta_\infty})}, \\ g\overline{g} = \dfrac{(f - q^{+\theta_t} t)(f - q^{-\theta_t} t)}{q(f - q^{+\theta_1})(f - q^{-\theta_1})}, \end{cases}$$

where

- $f, g : T \to \mathbb{CP}^1$ are complex functions on a discrete time domain

  $$T = q^{\mathbb{Z}} t_0 := \{\dots, q^{+2} t_0, q^{+1} t_0, t_0, q^{-1} t_0, q^{-2} t_0, \dots\},$$

  and t varies in this domain.
- $f = f(t)$, $\overline{f} = f(qt)$, and similar for g,
- $(\theta_0, \theta_t, \theta_1, \theta_\infty) \in \mathbb{C}^4$ and $t_0 \in \mathbb{C}^*$ are complex parameters.
- As $q \uparrow 1$ with $f \to u$ and $g \to (u - t)/(u - 1)$, qP_{VI} degenerates to P_{VI}.

Discovery of q-Painlevé VI

- Hilbert (1900) posed 21st problem on construction of a Fuchsian ODE with given monodromy group.
- Plemelj (1908) published (partially faulty) solution to Hilbert's 21st problem.
- Birkhoff (1913) extended Plemelj's work to include irregular singularities and also formulated and solved the analogs of Hilbert's 21st problem for Fuchsian difference and q-difference equations.
- Jimbo and Sakai (1996) considered a q-analog of R. Fuchs' problem in the framework developed by Birkhoff that lead to $q\mathrm{P}_{\mathrm{VI}}$.

Linear problem for q-Painlevé VI

Jimbo and Sakai (1996) derived $q\mathrm{P}_{\mathrm{VI}}$ by considering the linear problem

$$Y(qz) = A(z,t)Y(z), \qquad A(z,t) = A_0 + z\,A_1 + z^2 A_2,$$

where

$$A_0 = H\begin{bmatrix} q^{+\theta_0}t & 0 \\ 0 & q^{-\theta_0}t \end{bmatrix}H^{-1}, \qquad A_2 = \begin{bmatrix} q^{-\theta_\infty} & 0 \\ 0 & q^{+\theta_\infty} \end{bmatrix},$$

and

$$|A(z,t)| = \left(z - q^{+\theta_t}t\right)\left(z - q^{-\theta_t}t\right)\left(z - q^{+\theta_1}\right)\left(z - q^{-\theta_1}\right).$$

For fixed t, $\dim\{A(z,t)\} = 2 + 1$.

Parametrisation in terms of (f,g) and auxiliary variable w by

$$A_{12}(z,t) = q^{\theta_\infty} w(z - f),$$
$$A_{22}(f,t) = q(f - q^{+\theta_1})(f - q^{-\theta_1})g.$$

q-'Frobenius' solutions

A_0 and A_2 are invertible $\implies$ $z = 0$ and $z = \infty$ are **Fuchsian**.

Carmichael's (1912) general existence theorems yield canonical convergent series solutions around $z = 0$ and $z = \infty$,

$$Y_0(z,t) = \Psi_0(z,t) z^{\log_q(t)+\theta_0\sigma_3}, \qquad \Psi_0(z,t) = H(t) + \sum_{n=1}^{\infty} z^n M_n(t),$$

$$Y_\infty(z,t) = \Psi_\infty(z,t) z^{\log_q(z/q)-\theta_\infty\sigma_3}, \quad \Psi_\infty(z,t) = I + \sum_{n=1}^{\infty} z^{-n} N_n(t).$$

where $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Monodromy of linear problem is encapsulated by the single **connection matrix**

$$P(z,t) := Y_0(z,t)^{-1} Y_\infty(z,t).$$

Note: $P(qz,t) = P(z,t)$.

Lax pair

Theorem (Jimbo and Sakai 1996)

The connection matrix is invariant under $t \mapsto qt$, that is

$$P(z, qt) = P(z, t),$$

if and only if both Y_0 and Y_∞ satisfy

$$Y(z, qt) = B(z, t)Y(z, t),$$

for some rational matrix function $B(z, t)$.
Furthermore, in such case, B takes the form

$$B(z, t) = \frac{z^2 I + z\, B_0(t)}{(z - q^{1+\theta_t} t)(z - q^{1-\theta_t} t)}.$$

Compatibility of

$$Y(qz, t) = A(z, t)Y(z, t),$$
$$Y(z, qt) = B(z, t)Y(z, t),$$

is equivalent to $q\mathrm{P}_{\mathrm{VI}}$ and an auxiliary q-difference equation for w.

Monodromy manifold

The connection matrix $P(z,t)$ factorises as

$$P(z,t) = z^{\log_q(z/qt)} z^{-\theta_0\sigma_3} C(z,t) z^{-\theta_\infty\sigma_3}, \quad C(z,t) := \Psi_0(z,t)^{-1}\Psi_\infty(z,t),$$

where $C(z,t)$ has the following properties.

(1) $C(z,t)$ is analytic in $z \in \mathbb{C}^*$.

(2) $C(qz,t) = \frac{t}{z^2}\begin{bmatrix} q^{+\theta_0} & 0 \\ 0 & q^{-\theta_0} \end{bmatrix} C(z,t) \begin{bmatrix} q^{+\theta_\infty} & 0 \\ 0 & q^{-\theta_\infty} \end{bmatrix}$.

(3) $|C(z,t)| = \text{constant}(t) \times \theta_q(q^{-\theta_t}\frac{z}{t})\theta_q(q^{+\theta_t}\frac{z}{t})\theta_q(q^{-\theta_1}z)\theta_q(q^{+\theta_1}z)$.

(4) $C(z,qt) = z\,C(z,t)$.

Here $\theta_q(\cdot)$ is Jacobi's q-theta function

$$\theta_q(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} z^n.$$

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(4) $C(z,qt) = z\, C(z,t)$.

For fixed t, define the **monodromy manifold** $\mathcal{M}_q(\theta,t)$ as the space of matrices $C(z)$ satisfying (1)-(3), quotiented by arbitrary left and right-multiplication by diagonal matrices.

Ohyama, Ramis Sauloy ('20) first defined this monodromy manifold and studied some of its algebraic properties. They further derived so called **Mano decompositions** of its elements.

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(2) $C(qz,t) = \frac{t}{z^2}\begin{bmatrix} q^{+\theta_0} & 0 \\ 0 & q^{-\theta_0}\end{bmatrix} C(z,t) \begin{bmatrix} q^{+\theta_\infty} & 0 \\ 0 & q^{-\theta_\infty}\end{bmatrix}$.

(3) $|C(z,t)| = \text{constant}(t) \times \theta_q(q^{-\theta_t}\frac{z}{t})\theta_q(q^{+\theta_t}\frac{z}{t})\theta_q(q^{-\theta_1}z)\theta_q(q^{+\theta_1}z)$.

(4) $C(z,qt) = z\, C(z,t)$.

For fixed t, define the **monodromy manifold** $\mathcal{M}_q(\theta,t)$ as the space of matrices $C(z)$ satisfying (1)-(3), quotiented by arbitrary left and right-multiplication by diagonal matrices.

Ohyama, Ramis Sauloy ('20) first defined this monodromy manifold and studied some of its algebraic properties. They further derived so called **Mano decompositions** of its elements.

Tyurin parameters

For any 2×2 matrix R of rank 1, define $\pi(R) \in \mathbb{CP}^1$ by

$$R_1 = \pi(R) R_2, \quad R = (R_1, R_2).$$

To construct integrals of motion, we use **Tyurin parameters** of the connection matrix,

$$\rho_k = \pi(C(x_k)) \quad (1 \le k \le 4), \quad (x_1, x_2, x_3, x_4) = (q^{+\theta_t} t, q^{-\theta_t} t, q^{+\theta_1}, q^{-\theta_1}),$$

The Tyurin parameters satisfy

$$T_{12}\rho_1\rho_2 + T_{13}\rho_1\rho_3 + T_{14}\rho_1\rho_4 + T_{23}\rho_2\rho_3 + T_{24}\rho_2\rho_4 + T_{34}\rho_3\rho_4 = 0,$$
$$T'_{12}\rho_1\rho_2 + T'_{13}\rho_1\rho_3 + T'_{14}\rho_1\rho_4 + T'_{23}\rho_2\rho_3 + T'_{24}\rho_2\rho_4 + T'_{34}\rho_3\rho_4 \neq 0,$$

where, for any labeling $\{i, j, k, l\} = \{1, 2, 3, 4\}$,

$$T_{ij} = \frac{x_2 x_4}{q^{\theta_0+\theta_\infty} t} x_i x_l \theta_q\left(\frac{x_i}{x_j}, \frac{x_k}{x_l}, \frac{x_i x_j}{q^{+\theta_0-\theta_\infty} t}, \frac{x_k x_l}{q^{+\theta_0+\theta_\infty} t}\right), \quad T'_{ij} = T_{ij}|_{\theta_0=0}.$$

Integrals of motion

For any $1 \leq i < j \leq 4$,

$$\eta_{ij} = \frac{T_{ij}\rho_i\rho_j}{T'_{12}\rho_1\rho_2 + T'_{13}\rho_1\rho_3 + T'_{14}\rho_1\rho_4 + T'_{23}\rho_2\rho_3 + T'_{24}\rho_2\rho_4 + T'_{34}\rho_3\rho_4},$$

defines an **integral of motion** of $q\mathrm{P}_{\mathrm{VI}}$.

Theorem (Joshi and PR (2022))

The six integrals of motion,

$$\eta = (\eta_{12}, \eta_{13}, \eta_{14}, \eta_{23}, \eta_{24}, \eta_{34}),$$

lie on an explicit affine algebraic surface $\mathcal{F}_q$, see next slide.
The induced mapping

$$\mathrm{RH}_q(\theta, t_0) : \{\textit{Solutions of } qP_{VI}\} \to \mathcal{F}_q, (f, g) \mapsto \eta,$$

is a one-to-one correspondence, for generic values of the parameters.

In fact, RH_q is a **biholomorphism** when identifying the solution space of $q\mathrm{P}_{\mathrm{VI}}$ with the initial value space at t_0 [PR 2023].

An affine algebraic surface

The affine algebraic surface $\mathcal{F}_q$ is defined in $\{\eta \in \mathbb{C}^6\}$ by four equations,

$$\eta_{12} + \eta_{13} + \eta_{14} + \eta_{23} + \eta_{23} + \eta_{34} = 0,$$
$$a_{12}\eta_{12} + a_{13}\eta_{13} + a_{14}\eta_{14} + a_{23}\eta_{23} + a_{24}\eta_{23} + a_{34}\eta_{34} + 1 = 0,$$
$$\eta_{13}\eta_{23} - b_1\eta_{12}\eta_{34} = 0,$$
$$\eta_{14}\eta_{23} - b_2\eta_{12}\eta_{34} = 0,$$

where

$$a_{12} = \prod_{\epsilon=\pm1} \frac{\theta_q\left(q^{+\theta_\infty} t_0\right)}{\theta_q\left(q^{\epsilon\theta_0+\theta_\infty} t_0\right)}, \qquad a_{34} = \prod_{\epsilon=\pm1} \frac{\theta_q\left(q^{-\theta_\infty} t_0\right)}{\theta_q\left(q^{\epsilon\theta_0-\theta_\infty} t_0\right)},$$
$$a_{13} = \prod_{\epsilon=\pm1} \frac{\vartheta_q\left(\theta_t + \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\theta_0 + \theta_t + \theta_1 + \theta_\infty\right)}, \quad a_{24} = \prod_{\epsilon=\pm1} \frac{\vartheta_q\left(-\theta_t - \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\theta_0 - \theta_t - \theta_1 + \theta_\infty\right)},$$
$$a_{14} = \prod_{\epsilon=\pm1} \frac{\vartheta_q\left(\theta_t - \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\theta_0 + \theta_t - \theta_1 + \theta_\infty\right)}, \quad a_{23} = \prod_{\epsilon=\pm1} \frac{\vartheta_q\left(-\theta_t + \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\theta_0 - \theta_t + \theta_1 + \theta_\infty\right)},$$

and similar expressions for b_1, b_2, where $\vartheta_q(x) = \theta_q(q^x)$.
Ref: Joshi, PR - CMP '23

Segre surfaces

The surface $\mathcal{F}_q$ is an affine **Segre surface**.

A Segre surface is by definition the intersection of two quadrics in $\mathbb{CP}^4$,

$$\{\eta \in \mathbb{CP}^4 : P(\eta) = 0\} \cap \{\eta \in \mathbb{CP}^4 : Q(\eta) = 0\},$$

where P and Q quadratic polynomials.
They were introduced and studied by Corrado Segre (1884).

Generic Asymptotics

Theorem (PR '23)

Take a generic $\boldsymbol{\eta} \in \mathcal{F}_q$*, then the corresponding solution* (f, g) *of* qP_{VI} *admits simultaneous complete asymptotic expansions,*

$$f(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} F_{n,k} r_{0t}^{k} (-t)^{n+2k\sigma_{0t}},$$

$$g(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} G_{n,k} r_{0t}^{k} (-t)^{n+2k\sigma_{0t}},$$

absolutely convergent for small enough $t \in q^{\mathbb{Z}} t_0$*, and*

$$\frac{f(t)}{t} = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \dot{F}_{n,k} r_{01}^{k} (-t)^{-(n+2k\sigma_{01})},$$

$$\frac{1}{g(t)} = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \dot{G}_{n,k} r_{01}^{k} (-t)^{-(n+2k\sigma_{01})},$$

absolutely convergent for large enough $t \in q^{\mathbb{Z}} t_0$*, with integration constants* $\{\sigma_{0t}, r_{0t}\}$ *and* $\{\sigma_{01}, r_{01}\}$ *as explicit functions of* $\boldsymbol{\eta}$*.*

Some explicit formulas

The exponents are defined through

$$\frac{\vartheta_q(\sigma_{0t}-\theta_1+\theta_\infty)\vartheta_q(\sigma_{0t}+\theta_1-\theta_\infty)}{\vartheta_q(\sigma_{0t}+\theta_1+\theta_\infty)\vartheta_q(\sigma_{0t}-\theta_1-\theta_\infty)}=\frac{T_{14}\eta_{13}}{T_{13}\eta_{14}},$$

$$\frac{\vartheta_q(\sigma_{01}-\theta_t+\theta_\infty)\vartheta_q(\sigma_{01}+\theta_t-\theta_\infty)}{\vartheta_q(\sigma_{01}+\theta_t+\theta_\infty)\vartheta_q(\sigma_{01}-\theta_t-\theta_\infty)}=\frac{T_{23}\eta_{13}}{T_{13}\eta_{23}},$$

$$0<\Re\sigma_{0t},\Re\sigma_{01}<\tfrac{1}{2},$$

and

$$r_{0t}=c_{0t}\times s_{0t},\qquad s_{0t}=M_{0t}\left(\frac{T_{34}\eta_{23}}{T_{23}\eta_{34}}\right),$$

$$r_{01}=c_{01}\times s_{01},\qquad s_{01}=M_{01}\left(\frac{T_{34}\eta_{23}}{T_{23}\eta_{34}}\right),$$

where $M_{0t}(\cdot)$ and $M_{01}(\cdot)$ are some explicit Möbius transforms and

$$c_{0t}=\frac{\Gamma_q(1-2\sigma_{0t})^2}{\Gamma_q(1+2\sigma_{0t})^2}\prod_{\epsilon=\pm1}\frac{\Gamma_q(1+\theta_t+\epsilon\theta_0+\sigma_{0t})\Gamma_q(1+\theta_1+\epsilon\theta_\infty+\sigma_{0t})}{\Gamma_q(1+\theta_t+\epsilon\theta_0-\sigma_{0t})\Gamma_q(1+\theta_1+\epsilon\theta_\infty-\sigma_{0t})},$$

$$c_{01}=\frac{\Gamma_q(1-2\sigma_{01})^2}{\Gamma_q(1+2\sigma_{01})^2}\prod_{\epsilon=\pm1}\frac{\Gamma_q(1+\theta_1+\epsilon\theta_0+\sigma_{01})\Gamma_q(1+\theta_t+\epsilon\theta_\infty+\sigma_{01})}{\Gamma_q(1+\theta_1+\epsilon\theta_0-\sigma_{01})\Gamma_q(1+\theta_t+\epsilon\theta_\infty-\sigma_{01})}.$$

Leading order coefficients

Leading order coefficients in asymptotic expansion near $t = 0$,

$$F_{1,\pm 1} = q^{-\theta_t} \frac{\left(q^{\theta_t+\theta_0\mp\sigma_{0t}} - 1\right)\left(q^{\theta_t-\theta_0\mp\sigma_{0t}} - 1\right)\left(q^{\theta_1+\theta_\infty\mp\sigma_{0t}} - 1\right)}{\left(q^{\theta_1+\theta_\infty\pm\sigma_{0t}} - 1\right)\left(q^{\sigma_{0t}} - q^{-\sigma_{0t}}\right)^2},$$

$$F_{1,0} = \frac{2\left(q^{\theta_t} + q^{-\theta_t}\right) - \left(q^{\theta_0} + q^{-\theta_0}\right)\left(q^{\sigma_{0t}} + q^{-\sigma_{0t}}\right)}{\left(q^{\sigma_{0t}} - q^{-\sigma_{0t}}\right)^2},$$

$$G_{1,0} = \frac{2\left(q^{\theta_0} + q^{-\theta_0}\right) - \left(q^{\theta_t} + q^{-\theta_t}\right)\left(q^{\sigma_{0t}} + q^{-\sigma_{0t}}\right)}{\left(q^{\sigma_{0t}} - q^{-\sigma_{0t}}\right)^2} q^{-1},$$

$$G_{1,\pm 1} = -q^{-1\mp\sigma_{0t}} F_{1,\pm 1},$$

and similar formulas near $t = \infty$, e.g.

$$\dot{F}_{1,\pm 1} = q^{-\theta_1} \frac{\left(q^{\theta_1+\theta_0\mp\sigma_{01}} - 1\right)\left(q^{\theta_1-\theta_0\mp\sigma_{01}} - 1\right)\left(q^{\theta_t+\theta_\infty\mp\sigma_{01}} - 1\right)}{\left(q^{\theta_t+\theta_\infty\pm\sigma_{01}} - 1\right)\left(q^{\sigma_{01}} - q^{-\sigma_{01}}\right)^2}.$$

A short history of asymptotic studies

- Mano ('10): generic leading order asymptotics near $t = 0$ and $t = \infty$ and an implicit relation between them.
- Jimbo, Nagoya and Sakai ('17): conjectural complete (and fully explicit) asymptotic expansion near $t = 0$ of the generic $q\mathrm{P}_{\mathrm{VI}}$ tau-function.
- PR ('23): complete asymptotic expansions near $t = 0$ and $t = \infty$ with explicit connection formulas.

Set up for main RHP

Fix $\theta \in \mathbb{C}^4$, $t_0 \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, such that **non-resonance conditions** hold,

$$t_0 \notin q^{\mathbb{Z} \pm (\theta_t + \theta_1)}, q^{\mathbb{Z} \pm (\theta_t - \theta_1)}, \qquad 2\theta_0, 2\theta_t, 2\theta_1, 2\theta_\infty \notin \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \frac{2\pi i}{\log q}.$$

Pick a generic $\boldsymbol{\eta} \in \mathcal{F}_q(\theta, t_0)$ and construct a corresponding connection matrix $C_0(z)$, so $[C_0(z)] \in \mathcal{M}_q(\theta, t_0)$.

For

$$t \in T := q^{\mathbb{Z}} t_0 = \{\ldots, q^{+2} t_0, q^{+1} t_0, t_0, q^{-1} t_0, q^{-2} t_0, \ldots\},$$

define

$$C(z, t) = z^m C_0(z), \qquad t = q^m t_0 \qquad (m \in \mathbb{Z}).$$

Main RHP

Main Riemann-Hilbert problem

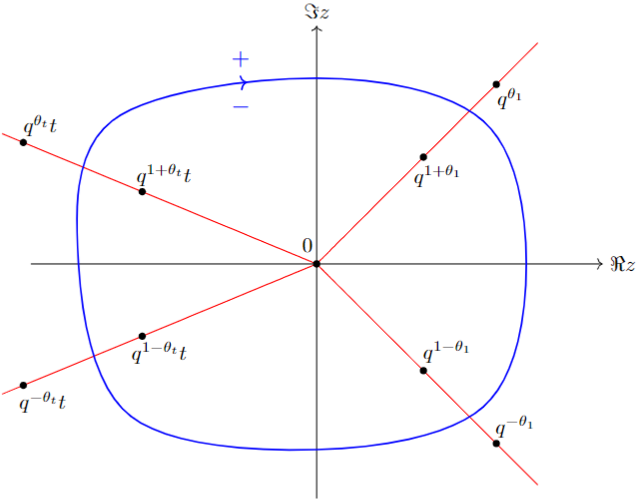
For $t \in T$ fixed, find 2×2 matrix-valued functions $\Psi_\infty(z,t)$ and $\Psi_0(z,t)$, which satisfy the following conditions with respect to z.

1. $\Psi_\infty(z,t)$ is analytic on $\mathbb{CP}^1 \setminus \{0\}$ and $\Psi_0(z,t)$ is analytic on
$$\mathbb{C} \setminus (q^{\mathbb{Z}_{\leq 0}} \cdot \{q^{\theta_t} t, q^{-\theta_t} t, q^{\theta_1}, q^{-\theta_1}\}).$$
2. $\Psi_\infty(z,t)$ and $\Psi_0(z,t)$ are related by
$$\Psi_\infty(z,t) = \Psi_0(z,t) C(z,t).$$
3. $\Psi_\infty(z,t)$ is normalised at infinity by
$$\Psi_\infty(z,t) = I + \mathcal{O}(z^{-1}) \qquad (z \to \infty).$$

For any $t \in T$, this RHP has at most one solution.

Contour for jump

Alternatively, RHP can be formulated with a single jump along a contour that moves with $t \in T$:

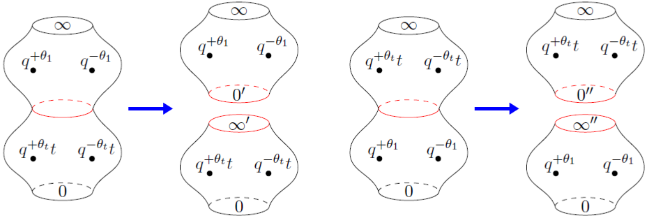


Strategy to obtain asymptotics around $t = 0$

We roughly follow strategy by Its, Lisovyy, Prokhorov (18') for $\mathrm{P_{VI}}$.

- Factorise global monodomy (i.e. connection matrix) into monodromies of two local q-hypergeometric systems, through **Mano-decomposition**.
- Factorise Riemann-Hilbert problem correspondingly.
- Construct **parametrices** which solve jumps of individual factors.
- Take quotient of global solution with respect to parametrices to arrive at a Riemann-Hilbert problem posed on a **circle**.
- Extract asymptotic expansions.

Mano-decompositions



(A) Decomposition I (B) Decomposition II

The connection matrix admits **Mano-decompositions**

$$C(z,t) = D_I(t) C_I^i(z/t)(-t)^{\boldsymbol{\sigma_{0t}}\sigma_3} \begin{bmatrix} r_{0t} & 0 \\ 0 & 1 \end{bmatrix} C_I^e(z),$$

$$= C_{II}^i(z)(-t)^{-\boldsymbol{\sigma_{01}}\sigma_3} \begin{bmatrix} r_{01} & 0 \\ 0 & 1 \end{bmatrix} C_{II}^e(z/t) D_{II}(t),$$

for $t \in T$, where $D_{I,II}(t)$ are some diagonal matrices and remaining components are connection matrices of q-hypergeometric systems.

Such decompositions were first observed in Mano's asymptotic study (2010) of $q\mathrm{P}_{\mathrm{VI}}$. Proven in general by Ohyama, Ramis and Sauloy (2020).

Decomposition I

Intermediate exponent: $\sigma_{0t} \in \mathbb{C}$, with $0 < \Re\sigma_{0t} < \frac{1}{2}$.
Twist parameter: $s_{0t} \in \mathbb{C}^*$, with $r_{0t} = c_{0t}s_{0t}$.

$$C_I^i(z) = \begin{bmatrix} c_{11}^i\,\theta_q(q^{-\sigma_{0t}-\theta_0}\,z) & c_{12}^i\,\theta_q(q^{+\sigma_{0t}-\theta_0}\,z) \\ c_{21}^i\,\theta_q(q^{-\sigma_{0t}+\theta_0}\,z) & c_{22}^i\,\theta_q(q^{+\sigma_{0t}+\theta_0}\,z) \end{bmatrix},$$

$$C_I^e(z) = \begin{bmatrix} c_{11}^e\,\theta_q(q^{-\theta_\infty+\sigma_{0t}}z) & c_{12}^e\,\theta_q(q^{+\theta_\infty+\sigma_{0t}}z) \\ c_{21}^e\,\theta_q(q^{-\theta_\infty-\sigma_{0t}}z) & c_{22}^e\,\theta_q(q^{+\theta_\infty-\sigma_{0t}}z) \end{bmatrix},$$

where the matrices of coefficients c^i and c^e read

$$c^i = \begin{bmatrix} \dfrac{\Gamma_q(+2\sigma_{0t},-2\theta_0)}{\Gamma_q(-\theta_t+\sigma_{0t}-\theta_0\,,1+\theta_t-\sigma_{0t}-\theta_0\,)} & \dfrac{\Gamma_q(-2\sigma_{0t},-2\theta_0)}{\Gamma_q(-\theta_t-\sigma_{0t}-\theta_0\,,1+\theta_t-\sigma_{0t}-\theta_0\,)} \\ \dfrac{\Gamma_q(+2\sigma_{0t},+2\theta_0)}{\Gamma_q(-\theta_t+\sigma_{0t}+\theta_0\,,1+\theta_t+\sigma_{0t}+\theta_0\,)} & \dfrac{\Gamma_q(-2\sigma_{0t},+2\theta_0)}{\Gamma_q(-\theta_t-\sigma_{0t}+\theta_0\,,1+\theta_t-\sigma_{0t}+\theta_0\,)} \end{bmatrix},$$

$$c^e = \begin{bmatrix} \dfrac{\Gamma_q(+2\theta_\infty,+2\sigma_{0t})}{\Gamma_q(-\theta_1+\theta_\infty+\sigma_{0t},1+\theta_1+\theta_\infty+\sigma_{0t})} & \dfrac{\Gamma_q(-2\theta_\infty,+2\sigma_{0t})}{\Gamma_q(-\theta_1-\theta_\infty+\sigma_{0t},1+\theta_1-\theta_\infty+\sigma_{0t})} \\ \dfrac{\Gamma_q(+2\theta_\infty,-2\sigma_{0t})}{\Gamma_q(-\theta_1+\theta_\infty-\sigma_{0t},1+\theta_1+\theta_\infty-\sigma_{0t})} & \dfrac{\Gamma_q(-2\theta_\infty,+2\sigma_{0t})}{\Gamma_q(-\theta_1-\theta_\infty-\sigma_{0t},1+\theta_1-\theta_\infty-\sigma_{0t})} \end{bmatrix}.$$

Decomposition I

Intermediate exponent: $\sigma_{0t} \in \mathbb{C}$, with $0 < \Re\sigma_{0t} < \frac{1}{2}$.
Twist parameter: $s_{0t} \in \mathbb{C}^*$, with $r_{0t} = c_{0t} s_{0t}$.

$$C_I^i(z) = \begin{bmatrix} c_{11}^i\, \theta_q(q^{-\sigma_{0t}-\theta_0}\, z) & c_{12}^i\, \theta_q(q^{+\sigma_{0t}-\theta_0}\, z) \\ c_{21}^i\, \theta_q(q^{-\sigma_{0t}+\theta_0}\, z) & c_{22}^i\, \theta_q(q^{+\sigma_{0t}+\theta_0}\, z) \end{bmatrix},$$

$$C_I^e(z) = \begin{bmatrix} c_{11}^e\, \theta_q(q^{-\theta_\infty+\sigma_{0t}} z) & c_{12}^e\, \theta_q(q^{+\theta_\infty+\sigma_{0t}} z) \\ c_{21}^e\, \theta_q(q^{-\theta_\infty-\sigma_{0t}} z) & c_{22}^e\, \theta_q(q^{+\theta_\infty-\sigma_{0t}} z) \end{bmatrix},$$

where the matrices of coefficients c^i and c^e read

$$c^i = \begin{bmatrix} \dfrac{\Gamma_q(+2\sigma_{0t}, -2\theta_0)}{\Gamma_q(-\theta_t + \sigma_{0t} - \theta_0\,, 1 + \theta_t - \sigma_{0t} - \theta_0\,)} & \dfrac{\Gamma_q(-2\sigma_{0t}, -2\theta_0)}{\Gamma_q(-\theta_t - \sigma_{0t} - \theta_0\,, 1 + \theta_t - \sigma_{0t} - \theta_0\,)} \\ \dfrac{\Gamma_q(+2\sigma_{0t}, +2\theta_0)}{\Gamma_q(-\theta_t + \sigma_{0t} + \theta_0\,, 1 + \theta_t + \sigma_{0t} + \theta_0\,)} & \dfrac{\Gamma_q(-2\sigma_{0t}, +2\theta_0)}{\Gamma_q(-\theta_t - \sigma_{0t} + \theta_0\,, 1 + \theta_t - \sigma_{0t} + \theta_0\,)} \end{bmatrix},$$

$$c^e = \begin{bmatrix} \dfrac{\Gamma_q(+2\theta_\infty, +2\sigma_{0t})}{\Gamma_q(-\theta_1 + \theta_\infty + \sigma_{0t}, 1 + \theta_1 + \theta_\infty + \sigma_{0t})} & \dfrac{\Gamma_q(-2\theta_\infty, +2\sigma_{0t})}{\Gamma_q(-\theta_1 - \theta_\infty + \sigma_{0t}, 1 + \theta_1 - \theta_\infty + \sigma_{0t})} \\ \dfrac{\Gamma_q(+2\theta_\infty, -2\sigma_{0t})}{\Gamma_q(-\theta_1 + \theta_\infty - \sigma_{0t}, 1 + \theta_1 + \theta_\infty - \sigma_{0t})} & \dfrac{\Gamma_q(-2\theta_\infty, +2\sigma_{0t})}{\Gamma_q(-\theta_1 - \theta_\infty - \sigma_{0t}, 1 + \theta_1 - \theta_\infty - \sigma_{0t})} \end{bmatrix}.$$

Factorised RHP

Find piece-wise analytic matrix function $\Psi(z,t)$ with jumps:

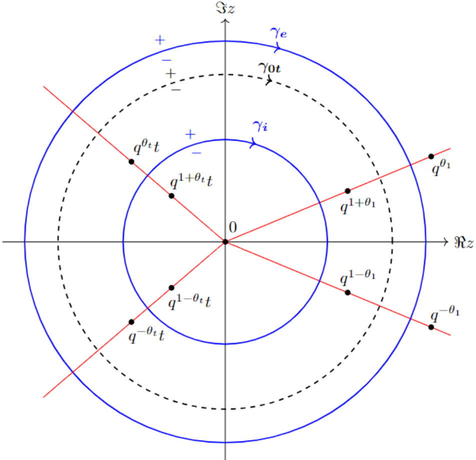
$$\Psi_+(z,t) = \Psi_-(z,t)C_I^e(z), \qquad (z \in \gamma_e),$$

$$\Psi_+(z,t) = \Psi_-(z,t)D(t)C_I^i\left(\frac{z}{t}\right)(-t)^{\sigma_{0t}\sigma_3}\begin{bmatrix} r_{0t} & 0 \\ 0 & 1 \end{bmatrix}, \qquad (z \in \gamma_i).$$

and normalisation

$$\Psi(z,t) = I + \mathcal{O}(z^{-1}),$$

as $z \to \infty$.



Parametrices

Individual jumps can be solved in terms of parametrices made out of Heine hypergeometric functions. For example, we can construct explicit $\Psi_\infty^e(z)$ and $\Psi_0^e(z)$ with

$$\Psi_\infty^e(z) = \Psi_0^e(z) C_I^e(z),$$

where

$$Y_\infty^e(z) := \Psi_\infty^e(z) z^{\frac{1}{2}\log_q(z/q) - \theta_\infty \sigma_3},$$
$$Y_0^e(z) := \Psi_\infty^e(z) z^{\log_q(-1) - \sigma_{0t}\sigma_3},$$

define solutions to q-hypergeometric system

$$Y(qz) = A^e(z) Y(z), \quad A^e(z) = \begin{bmatrix} \alpha^e & \beta^e \\ \gamma^e & \delta^e \end{bmatrix} + z \begin{bmatrix} q^{-\theta_\infty} & 0 \\ 0 & q^{+\theta_\infty} \end{bmatrix},$$

characterised by

$$A^e(0) \sim \begin{bmatrix} -q^{-\sigma_{0t}} & 0 \\ 0 & -q^{+\sigma_{0t}} \end{bmatrix}, \quad |A^e(z)| = (z - q^{+\theta_1})(z - q^{-\theta_1}).$$

Coefficients of q-hypergeometric system

$$
\begin{aligned}
\alpha^e &= \frac{q^{\theta_1} + q^{-\theta_1} - (q^{-\theta_\infty + \sigma_{0t}} + q^{-\theta_\infty - \sigma_{0t}})}{q^{-\theta_\infty} - q^{+\theta_\infty}}, \\
\beta^e &= \frac{(q^{\theta_1 + \theta_\infty + \sigma_{0t}} - 1)(q^{\theta_1 + \theta_\infty - \sigma_{0t}} - 1)}{q^{\theta_1}(q^{+\theta_\infty} - q^{-\theta_\infty})}, \\
\gamma^e &= \frac{(q^{\theta_1 - \theta_\infty + \sigma_{0t}} - 1)(q^{\theta_1 - \theta_\infty - \sigma_{0t}} - 1)}{q^{\theta_1}(q^{-\theta_\infty} - q^{+\theta_\infty})}, \\
\delta^e &= \frac{q^{\theta_1} + q^{-\theta_1} - (q^{+\theta_\infty + \sigma_{0t}} + q^{+\theta_\infty - \sigma_{0t}})}{q^{+\theta_\infty} - q^{-\theta_\infty}}.
\end{aligned}
$$

Explicit formula for part of parametrix

$$\Psi^e_\infty(z) = \widehat{\Psi}^e_\infty(z) \begin{pmatrix} \left(q^{1+\theta_1}/z;q\right)_\infty & 0 \\ 0 & \left(q^{1-\theta_1}/z;q\right)_\infty \end{pmatrix},$$

where $(z;q)_\infty = (1-z)(1-qz)(1-q^2z)\dots$ and

$$\widehat{\Psi}^e_{\infty,11}(z) = {}_2\phi_1\left[\begin{matrix} q^{-\theta_1+\theta_\infty+\sigma_{0t}}, q^{-\theta_1+\theta_\infty-\sigma_{0t}} \\ q^{2\theta_\infty} \end{matrix}; q, \frac{q^{1+\theta_1}}{z}\right],$$

$$\widehat{\Psi}^e_{\infty,12}(z) = \frac{r_1^e}{z}\, {}_2\phi_1\left[\begin{matrix} q^{1+\theta_1-\theta_\infty+\sigma_{0t}}, q^{1+\theta_1-\theta_\infty-\sigma_{0t}} \\ q^{2-2\theta_\infty} \end{matrix}; q, \frac{q^{1-\theta_1}}{z}\right],$$

$$\widehat{\Psi}^e_{\infty,21}(z) = \frac{r_2^e}{z}\, {}_2\phi_1\left[\begin{matrix} q^{1-\theta_1+\theta_\infty+\sigma_{0t}}, q^{1-\theta_1+\theta_\infty-\sigma_{0t}} \\ q^{2+2\theta_\infty} \end{matrix}; q, \frac{q^{1+\theta_1}}{z}\right],$$

$$\widehat{\Psi}^e_{\infty,22}(z) = {}_2\phi_1\left[\begin{matrix} q^{\theta_1-\theta_\infty+\sigma_{0t}}, q^{\theta_1-\theta_\infty-\sigma_{0t}} \\ q^{-2\theta_\infty} \end{matrix}; q, \frac{q^{1-\theta_1}}{z}\right],$$

with

$$r_1^e = \frac{q\,\beta^e}{q^{\theta_\infty} - q^{-\theta_\infty}}, \qquad r_2^e = \frac{q\,\gamma^e}{q^{-\theta_\infty} - q^{1+\theta_\infty}}.$$

RHP on a circle

Find piece-wise analytic matrix function $\Phi(z,t)$ with jump

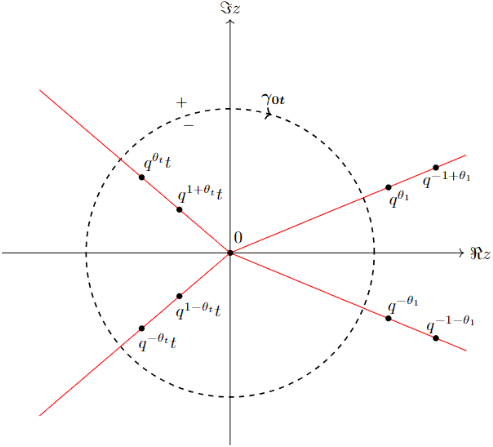
$$\Phi_+(z,t) = \Phi_-(z,t)J(z,t), \quad (z \in \gamma_{0t}),$$

$$J(z,t) := \Psi_0^e(z)\begin{bmatrix} 1/r_{0t} & 0 \\ 0 & 1 \end{bmatrix}(-t)^{-\sigma_{0t}\sigma_3}\Psi_\infty^i(z/t)(-t)^{+\sigma_{0t}\sigma_3}\begin{bmatrix} r_{0t} & 0 \\ 0 & 1 \end{bmatrix}\Psi_0^e(z)^{-1},$$

and normalisation

$$\Phi(z,t) = I + \mathcal{O}(z^{-1}),$$

as $z \to \infty$.



Jump matrix expansion

The jump matrix is a **perturbation of the identity matrix** for small t,

$$J(z,t) = I + \sum_{n=1}^{\infty} r_{0t}(-t)^{n+2\sigma_{0t}} J_n^+(z) + (-t)^n J_n^0(z) + 1/r_{0t}(-t)^{n-2\sigma_{0t}} J_n^-(z),$$

where $J_n^-(z), J_n^0(z) J_n^+(z)$ are analytic in a neighbourhood of γ_{0t}, $n \geq 1$.

Using standard Riemann-Hilbert machinery, it follows that the RHP for $\Phi(z,t)$ is solvable for small enough $t \in T$, and the solution admits an expansion

$$\Phi(z,t) = I + \sum_{n=1}^{\infty} \sum_{k=-n}^{n} r_{0t}^{k} (-t)^{n+2k\sigma_{0t}} \Phi_{n,k}(z),$$

which is uniformly absolutely convergent in $z \in \mathbb{CP}^1 \setminus \gamma_{0t}$ with respect to the max norm.

Generic Asymptotics

Theorem (PR '23)

Take a generic $\boldsymbol{\eta} \in \mathcal{F}_q$, then the corresponding solution (f, g) of qP_{VI} admits simultaneous complete asymptotic expansions,

$$f(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} F_{n,k} r_{0t}^{k} (-t)^{n+2k\sigma_{0t}},$$
$$g(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} G_{n,k} r_{0t}^{k} (-t)^{n+2k\sigma_{0t}},$$

absolutely convergent for small enough $t \in q^{\mathbb{Z}} t_0$, and

$$\frac{f(t)}{t} = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \dot{F}_{n,k} r_{01}^{k} (-t)^{-(n+2k\sigma_{01})},$$
$$\frac{1}{g(t)} = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \dot{G}_{n,k} r_{01}^{k} (-t)^{-(n+2k\sigma_{01})},$$

absolutely convergent for large enough $t \in q^{\mathbb{Z}} t_0$, with integration constants $\{\sigma_{0t}, r_{0t}\}$ and $\{\sigma_{01}, r_{01}\}$ as explicit functions of $\boldsymbol{\eta}$.

Lines on Segre surface

Theorem (Corrado Segre (1884))

A smooth Segre surface contains exactly **16 lines**.

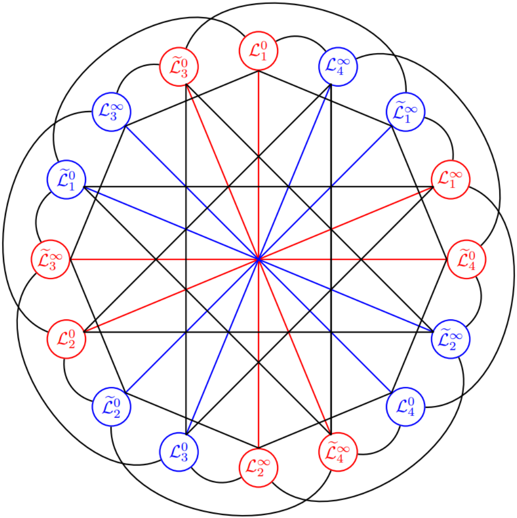
Within $\mathcal{F}_q$, lines are given by

$$\begin{aligned} \mathcal{L}_k^0 &= \{\eta \in \mathcal{F}_q : \rho_k = 0\}, \\ \mathcal{L}_k^\infty &= \{\eta \in \mathcal{F}_q : \rho_k = \infty\}, \\ \widetilde{\mathcal{L}}_k^0 &= \{\eta \in \mathcal{F}_q : \widetilde{\rho}_k = 0\}, \\ \widetilde{\mathcal{L}}_k^\infty &= \{\eta \in \mathcal{F}_q : \widetilde{\rho}_k = \infty\}, \end{aligned}$$

for $1 \le k \le 4$, where $\widetilde{\rho}_k = \pi(C(x_k)^T)$.
Explicitly, e.g.

$$\begin{aligned} \mathcal{L}_1^0 =\{\eta \in \mathbb{C}^6 : \eta_{12} = \eta_{13} = \eta_{14} = 0, \eta_{23} + \eta_{24} + \eta_{34} = 0, \\ a_{23}\eta_{23} + a_{24}\eta_{24} + a_{34}\eta_{34} + 1 = 0\}. \end{aligned}$$

Intersection graph of lines



- vertices : lines
- edges : intersection points

Lines and reducible factors

Recall **Mano-decompositions**

$$C(z,t) = D_I(t)C_I^i(z/t)(-t)^{\boldsymbol{\sigma_{0t}}\sigma_3}\begin{bmatrix} r_{0t} & 0 \\ 0 & 1 \end{bmatrix} C_I^e(z),$$

$$= C_{II}^i(z)(-t)^{-\boldsymbol{\sigma_{01}}\sigma_3}\begin{bmatrix} r_{01} & 0 \\ 0 & 1 \end{bmatrix} C_{II}^e(z/t)D_{II}(t).$$

- On blue lines, one of the factors in decomposition I is **reducible**, i.e. $C_I^i(z)$ or $C_I^e(z)$ is triangular or anti-triangular.
- On red lines, one of the factors in decomposition II is **reducible**, i.e. $C_{II}^i(z)$ or $C_{II}^e(z)$ is triangular or anti-triangular.

Truncation on lines

On the blue lines, generic asymptotics near $t=0$ **truncate**.
For example, on the line $\widetilde{\mathcal{L}}_2^\infty$, we have $\sigma_{0t}=\theta_t-\theta_0$, and

$$f(t)=\sum_{n=1}^{\infty}\sum_{k=-n}^{0}F_{n,k}r_{0t}^{k}(-t)^{n+2k(\theta_t-\theta_0)},$$

$$g(t)=\sum_{n=1}^{\infty}\sum_{k=-n}^{0}G_{n,k}r_{0t}^{k}(-t)^{n+2k(\theta_t-\theta_0)},$$

if $\Re(\theta_t-\theta_0)<\frac{1}{2}$.
On the **intersection point** of blue lines $\widetilde{\mathcal{L}}_2^\infty$ and $\widetilde{\mathcal{L}}_1^0$, we have $r_{0t}=0$ and generic asymptotics are **doubly truncated**,

$$f(t)=\sum_{n=1}^{\infty}F_{n,0}(-t)^n, \qquad F_{1,0}=\frac{q^{\theta_0}-q^{-\theta_0}}{q^{\theta_t-\theta_0}-q^{\theta_0-\theta_t}},$$

$$g(t)=\sum_{n=1}^{\infty}G_{n,0}(-t)^n, \qquad G_{1,0}=\frac{q^{\theta_t}-q^{-\theta_t}}{q^{\theta_0-\theta_t}-q^{\theta_t-\theta_0}}.$$

The latter power series solutions should be called Kaneko-Ohyama solutions (2013,2015).

black intersection points

Let (f,g) be the solution corresponding to the intersection point

$$\{\eta_*(t)\} = \widetilde{\mathcal{L}}_1^0 \cap \widetilde{\mathcal{L}}_3^\infty,$$

and assume $\Re(\theta_0 - \theta_t), \Re(-\theta_0 - \theta_1) < \frac{1}{2}$, then $f(t)$ admits simultaneous uniformly convergent asymptotic expansions

$$f(t) = \frac{q^{\theta_0} - q^{-\theta_0}}{q^{\theta_0-\theta_t} - q^{\theta_t-\theta_0}}\, t + tE_0(t) + \sum_{n=2}^{\infty}\sum_{k=0}^{n} f_{n,k} t^n E_0(t)^k \qquad (t \to 0),$$

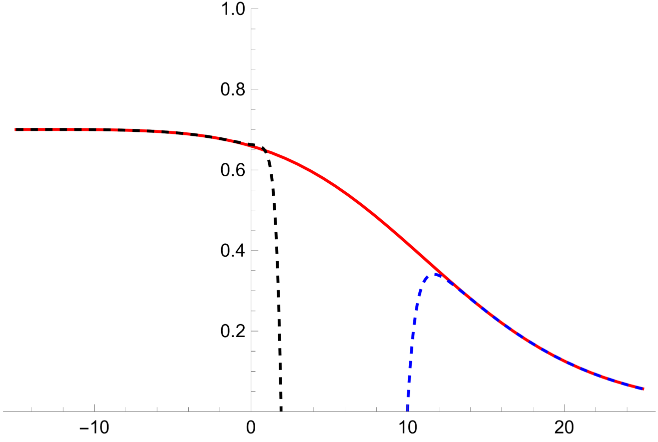
$$f(t) = \frac{q^{\theta_0} - q^{-\theta_0}}{q^{\theta_0+\theta_1} - q^{-\theta_0-\theta_1}} + E_\infty(t) + t^{-1}\sum_{n=2}^{\infty}\sum_{k=0}^{n} \dot{f}_{n,k} t^{-n} E_\infty(t)^k \quad (t \to \infty),$$

on compact sets $K \subseteq \mathbb{CP}^1 \setminus q^{\mathbb{Z}-2\theta_0+\theta_t-\theta_1}$, with $qK = K$, where

$$E_0(t) = c_0 \frac{\theta_q(q^{-\theta_t-\theta_1} t)}{\theta_q(q^{-2\theta_0+\theta_t-\theta_1} t)}, \qquad E_\infty(t) = c_\infty \frac{\theta_q(q^{-\theta_t-\theta_1} t^{-1})}{\theta_q(q^{+2\theta_0-\theta_t+\theta_1} t^{-1})},$$

for some explicit constant factors c_0, c_∞.

Plot of f on negative real line



Plot of $f(-q^r)$ in red with $r \in (-15, 25)$ and parameter values

$$\theta_0 = \tfrac{1}{3}, \quad \theta_t = \tfrac{1}{5}, \quad \theta_1 = \tfrac{1}{7}, \quad \theta_\infty = \tfrac{1}{11}, \quad q = \exp(-\tfrac{1}{4}).$$

In dashed black and blue the series expansions around $t = -\infty$ and $t = 0$ respectively.

Continuum limit

- $\mathcal{F}_q$ is a six-dimensional family of affine Segre surfaces.
- Taking the limit $q \uparrow 1$, $q\mathrm{P}_{\mathrm{VI}}$ reduces to the usual sixth Painlevé equation.
- The formulas for generic asymptotics near $t = 0$ and $t = \infty$ reduce to Jimbo's ('82) formulas for P_{VI} in this limit.
- The limit of the Segre surface is also well-defined,

$$\lim_{q \uparrow 1} \mathcal{F}_q = \mathcal{F}_1,$$

  and t_0-dependence drops out in limit.

Theorem (Joshi, Mazzocco, PR - to appear)

The affine Segre surface $\mathcal{F}_1$ is isomorphic to the affine cubic surface for P_{VI} with one line at infinity blown down.

q-Painlevé IV

Fix $0 < |q| < 1$, then $q\mathrm{P}_{\mathrm{IV}}$ is given by the coupled system

$$\frac{\overline{f}_0}{a_0 a_1 f_1} = \frac{1 + a_2 f_2 (1 + a_0 f_0)}{1 + a_0 f_0 (1 + a_1 f_1)},$$
$$\frac{\overline{f}_1}{a_1 a_2 f_2} = \frac{1 + a_0 f_0 (1 + a_1 f_1)}{1 + a_1 f_1 (1 + a_2 f_2)},$$
$$\frac{\overline{f}_2}{a_2 a_0 f_0} = \frac{1 + a_1 f_1 (1 + a_2 f_2)}{1 + a_2 f_2 (1 + a_0 f_0)},$$

where $f_k = f_k(t)$, $\overline{f}_k = f_k(qt)$, $k = 0, 1, 2$, and (a_0, a_1, a_2) are complex constants, subject to

$$f_0 f_1 f_2 = t^2, \qquad a_0 a_1 a_2 = q.$$

Symmetric form derived by Kajiwara, Noumi, Yamada (2000).

Monodromy manifold for $q\mathrm{P}_{\mathrm{IV}}$

Monodromy manifold for $q\mathrm{P}_{\mathrm{IV}}$ is given by hypersurface in $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$,

$$\begin{aligned} 0 = &+ \theta_q(+a_0)\theta_q(+a_1)\theta_q(+a_2)\left[\theta_q(t_0)\boldsymbol{\rho_1\rho_2\rho_3} - \theta_q(-t_0)\right] \\ &- \theta_q(-a_0)\theta_q(+a_1)\theta_q(-a_2)\left[\theta_q(t_0)\boldsymbol{\rho_1} - \theta_q(-t_0)\boldsymbol{\rho_2\rho_3}\right] \\ &+ \theta_q(+a_0)\theta_q(-a_1)\theta_q(-a_2)\left[\theta_q(t_0)\boldsymbol{\rho_2} - \theta_q(-t_0)\boldsymbol{\rho_1\rho_3}\right] \\ &- \theta_q(-a_0)\theta_q(-a_1)\theta_q(+a_2)\left[\theta_q(t_0)\boldsymbol{\rho_3} - \theta_q(-t_0)\boldsymbol{\rho_1\rho_2}\right], \end{aligned}$$

minus a curve, where $\theta_q(\cdot)$ is Jacobi's q-theta function

$$\theta_q(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} z^n.$$

See Joshi, PR - CMP (2021).

Which classical algebraic surface is this?

Affine Del Pezzo surfaces

Del Pezzo (1887) studied algebraic surfaces with a degree d embedding in $\mathbb{CP}^d$. They can all be obtained by blowing up $9 - d$ points in $\mathbb{CP}^2$ in general position, $3 \leq d \leq 9$.

- The monodromy manifolds of the classical Painlevé equations are affine cubic surfaces. These are affine Del Pezzo surfaces of degree **three**.
- The monodromy manifold of $q\mathrm{P}_{\mathrm{VI}}$ is an affine Segre surface, which is an affine Del Pezzo surface of degree **four**.
- The monodromy manifold of $q\mathrm{P}_{\mathrm{IV}}$ is an affine Del Pezzo surface of degree **six**.
- Are all the monodromy manifolds of Painlevé equations affine Del Pezzo surfaces?

Generic parameter values

Define the lattice

$$\Lambda_q := \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot \frac{2\pi i}{\log q},$$

then the **parameters** Θ, t_0, q are called **generic** if:

$$2\theta_0, 2\theta_t, 2\theta_1, 2\theta_\infty \notin \Lambda_q, \qquad \theta_0 \pm_1 \theta_t \pm_2 \theta_1 \pm_3 \theta_\infty \notin \Lambda_q,$$

and

$$t_0 \notin q^{\mathbb{Z}\pm(\theta_t+\theta_1)}, q^{\mathbb{Z}\pm(\theta_t-\theta_1)}, q^{\mathbb{Z}\pm(\theta_0+\theta_\infty)}, q^{\mathbb{Z}\pm(\theta_0-\theta_\infty)},$$

and

$$2\theta_0 \pm 2\theta_t, 2\theta_0 \pm 2\theta_1, 2\theta_\infty \pm 2\theta_t, 2\theta_\infty \pm 2\theta_1 \notin \Lambda_q,$$

and

$$t_0 \notin q^{\theta_0 \pm_1 \theta_\infty \pm_2 2\theta_t}, q^{\theta_0 \pm_1 \theta_\infty \pm_2 2\theta_1}, q^{\theta_t \pm_1 \theta_1 \pm_2 2\theta_0}, q^{\theta_t \pm_1 \theta_1 \pm_2 2\theta_\infty}.$$

blue : guarantee no resonance,

red : guarantee no globally reduced monodromy,

black : guarantee existence of simultaneously reduced factors in Mano-decompositions.