On q-Painlevé VI and the Geometry of Segre surfaces

Pieter Roffelsen

The University of Sydney

August, 2023

ICIAM 2023, Tokyo

Motivation

Painlevé equations are famous second order nonlinear difference/differential equations that are integrable.

What are the integrals of motion of Painlevé equations?

For the classical (differential) Painlevé equations, this question was answered during the 80s and 90s. But, what about the discrete Painlevé equations?

An answer for $q\mathrm{P}_{\mathrm{VI}}$ is obtained in:
N. Joshi and PR - *On the monodromy manifold of q-Painlevé VI and its Riemann-Hilbert Problem* (2022), to appear in CMP.

For $q\mathrm{P}_{\mathrm{VI}}$, the integrals of motion lie on a classical algebraic surface known as a **Segre surface**. Some consequences:
PR - *On q-Painlevé VI and the geometry of Segre surfaces* (2023).

Plan talk

1. Background on Painlevé VI
2. q-Painlevé VI and a family of Segre surfaces
3. Geometry and solutions
4. Outlook

Painlevé VI

- Painlevé VI, discovered by R. Fuchs (1905), is the most general second order nonlinear ODE without movable branch points or essential singularities.
- It is explicitly given by

$$u_{tt} = \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\frac{u_t^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u_t$$
$$+\frac{u(u-1)(u-t)}{2t^2(t-1)^2}\left((2\theta_\infty - 1)^2 - \frac{4\theta_0^2 t}{u^2} + \frac{4\theta_1^2(t-1)}{(u-1)^2} + \frac{(1-4\theta_t^2)t(t-1)}{(u-t)^2}\right),$$

  where $\Theta := (\theta_0, \theta_t, \theta_1, \theta_\infty) \in \mathbb{C}^4$ are complex parameters.
- Applications in: conformal field theory, general relativity, random matrix theory, topological field theory,. . . monodromy preserving deformations of certain linear ODEs.

Integrability

$P_{\rm VI}$ governs monodromy preserving deformations of rank two Fuchsian systems with four singularities,

$$Y_z = A(z,t)Y, \quad A(z,t) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}.$$

Monodromy of such an equation defines a point in the quotient space

$$\{(M_0, M_t, M_1) \in SL_2(\mathbb{C})^3\}//SL_2(\mathbb{C}).$$

Fricke and Klein (1897) showed that a point in this space is uniquely determined by the values of the trace coordinates

$$r_1 = \mathrm{Tr}\, M_0, \qquad r_2 = \mathrm{Tr}\, M_t, \qquad r_3 = \mathrm{Tr}\, M_1, \qquad r_4 = \mathrm{Tr}\, M_1 M_t M_0,$$
$$\eta_1 = \mathrm{Tr}\, M_0 M_t, \quad \eta_2 = \mathrm{Tr}\, M_0 M_1, \quad \eta_3 = \mathrm{Tr}\, M_t M_1,$$

which satisfy the single constraint

$$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + \eta_3^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + w_4 = 0,$$

where

$$w_1 = -(r_1 r_4 + r_2 r_3), \qquad w_2 = -(r_2 r_4 + r_1 r_3),$$
$$w_3 = -(r_3 r_4 + r_1 r_2), \qquad w_4 = r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_1 r_2 r_3 r_4 - 4.$$

Integrals of motion

The three trace coordinates

$$\eta_1 = \mathrm{Tr}\, M_0 M_t, \quad \eta_2 = \mathrm{Tr}\, M_0 M_1, \quad \eta_3 = \mathrm{Tr}\, M_t M_1,$$

generically form a complete set of first integrals of $\mathrm{P_{VI}}$, that lie on an **affine cubic surface**.

Jimbo (1982) first related $\mathrm{P_{VI}}$ with the cubic surface $\mathcal{F}$.

P-eqs	polynomials
P_{VI}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + \eta_3^2 + w_1\eta + w_2\eta_2 + w_3\eta_3 + w_4$
P_{V}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + R(w_{1,2,3})$
P_{IV}	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 + w_2(\eta_2 + \eta_3) + w_2(1 + w_1 - w_2)$
$P_{\mathrm{III}}^{D_6}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_1 - 1$
$\vdots$	$\vdots$
$P_{\mathrm{II}}^{\mathrm{JM}}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 - \eta_2 - 1$
P_{I}	$\eta_1\eta_2\eta_3 - \eta_1 - \eta_2 + 1$

See e.g. van der Put and Saito (2009) and Chekhov et al. (2015).

q-Painlevé VI

Let $q \in \mathbb{C}$ with $0 < |q| < 1$, then q-Painlevé VI is given by

$$q\mathrm{P}_{\mathrm{VI}}(\Theta, t_0) : \begin{cases} f\overline{f} = \dfrac{(\overline{g} - q^{+\theta_0} t)(\overline{g} - q^{-\theta_0} t)}{(\overline{g} - q^{\theta_\infty - 1})(\overline{g} - q^{-\theta_\infty})}, \\ g\overline{g} = \dfrac{(f - q^{+\theta_t} t)(f - q^{-\theta_t} t)}{q(f - q^{+\theta_1})(f - q^{-\theta_1})}, \end{cases}$$

where

- $f, g : T \to \mathbb{CP}^1$ are complex functions on a discrete time domain

  $$T = q^{\mathbb{Z}} t_0 := \{\ldots, q^{+2} t_0, q^{+1} t_0, t_0, q^{-1} t_0, q^{-2} t_0, \ldots\},$$

  and t varies in this domain.
- $f = f(t)$, $\overline{f} = f(qt)$, and similar for g,
- $\Theta = (\theta_0, \theta_t, \theta_1, \theta_\infty) \in \mathbb{C}^4$ and $t_0 \in \mathbb{C}^*$ are complex parameters.

Discovery of $q\mathrm{P}_{\mathrm{VI}}$

Jimbo and Sakai (1996) derived $q\mathrm{P}_{\mathrm{VI}}$ as governing deformation of a rank and degree two Fuchsian q-difference system which leaves monodromy invariant.

After normalisation, this linear system takes the form

$$Y(qz) = A(z,t)\,Y(z), \qquad A(z,t) = A_0 + z\,A_1 + z^2 A_2,$$

where

$$A_0 = H\,q^{\theta_0\sigma_3}\,t\,H^{-1}, \qquad A_2 = q^{-\theta_\infty\sigma_3}, \qquad \sigma_3 := \mathrm{diag}(1,-1),$$

for some $H = H(t) \in GL_2(\mathbb{C})$, and

$$|A(z,t)| = \left(z - q^{+\theta_t}t\right)\left(z - q^{-\theta_t}t\right)\left(z - q^{+\theta_1}\right)\left(z - q^{-\theta_1}\right).$$

Standard coordinates on space of such matrices is given by $\{f, g, w\}$, where

$$A_{12}(z,t) = q^{\theta_\infty}w(z-f), \quad A_{22}(f,t) = q(f - q^{+\theta_1})(f - q^{-\theta_1})g.$$

Monodromy

Following Birkhoff (1913) and Sauloy (2002), the **monodromy** of such a system is encapsulated by a single **connection matrix**,

$$C(z) := \Psi_0(z)^{-1}\Psi_\infty(z),$$

where $\Psi_0(z)$ and $\Psi_\infty(z)$ are meromorphic matrix functions on $\mathbb{CP}^1 \setminus \{\infty\}$ and $\mathbb{CP}^1 \setminus \{0\}$ resp. that define canonical solutions around $z = 0$ and $z = \infty$,

$$\begin{aligned} Y_0(z) &= \Psi_0(z) z^{\log_q(t)+\theta_0\sigma_3}, & \Psi_0(z) &= H + \mathcal{O}(z) & (z \to 0),\\ Y_\infty(z) &= \Psi_\infty(z) z^{\log_q(z/q)-\theta_\infty\sigma_3}, & \Psi_\infty(z) &= I + \mathcal{O}(z^{-1}) & (z \to \infty), \end{aligned}$$

of the linear system.

Monodromy manifold

The connection matrix $C(z)$ satisfies

(1) $C(z)$ is analytic on $\mathbb{C}^*$.

(2) $C(qz) = t\, z^{-2} q^{\theta_0 \sigma_3} C(z) q^{\theta_\infty \sigma_3}$.

(3) $|C(z)| = \text{constant} \times \theta_q(q^{-\theta_t} \frac{z}{t}) \theta_q(q^{+\theta_t} \frac{z}{t}) \theta_q(q^{-\theta_1} z) \theta_q(q^{+\theta_1} z)$,

where $\theta_q(\cdot)$ denotes the modified Jacobi theta function.

Define the **monodromy manifold** $\mathcal{M}_q(\Theta, t)$ as the space of matrices $C(z)$ satisfying (1)-(3), quotiented by arbitrary left and right-multiplication by diagonal matrices.

This space was first introduced and studied by Ohyama, Ramis and Sauloy (2020). They showed that it naturally comes with the structure of an algebraic variety and derived Mano-decompositions of its elements.

Tyurin parameters

For any 2×2 matrix R of rank 1, define $\pi(R) \in \mathbb{CP}^1$ by

$$R_1 = \pi(R) R_2, \quad R = (R_1, R_2).$$

To construct integrals of motion, we use **Tyurin parameters** of the connection matrix,

$$\rho_k = \pi(C(x_k)) \quad (1 \le k \le 4), \quad (x_1, x_2, x_3, x_4) = (q^{+\theta_t} t, q^{-\theta_t} t, q^{+\theta_1}, q^{-\theta_1}).$$

The Tyurin parameters satisfy

$$T_{12}\rho_1\rho_2 + T_{13}\rho_1\rho_3 + T_{14}\rho_1\rho_4 + T_{23}\rho_2\rho_3 + T_{24}\rho_2\rho_4 + T_{34}\rho_3\rho_4 = 0,$$
$$T'_{12}\rho_1\rho_2 + T'_{13}\rho_1\rho_3 + T'_{14}\rho_1\rho_4 + T'_{23}\rho_2\rho_3 + T'_{24}\rho_2\rho_4 + T'_{34}\rho_3\rho_4 \neq 0,$$

where, for any labeling $\{i, j, k, l\} = \{1, 2, 3, 4\}$,

$$T_{ij} = \frac{x_2 x_4}{q^{\theta_0 + \theta_\infty} t} x_i x_l \theta_q \left(\frac{x_i}{x_j}, \frac{x_k}{x_l}, \frac{x_i x_j}{q^{+\theta_0 - \theta_\infty} t}, \frac{x_k x_l}{q^{+\theta_0 + \theta_\infty} t} \right), \quad T'_{ij} = T_{ij}|_{\theta_0 = 0}.$$

Integrals of motion

For any $1 \leq i < j \leq 4$,

$$\eta_{ij} = \frac{T_{ij}\rho_i\rho_j}{T'_{12}\rho_1\rho_2 + T'_{13}\rho_1\rho_3 + T'_{14}\rho_1\rho_4 + T'_{23}\rho_2\rho_3 + T'_{24}\rho_2\rho_4 + T'_{34}\rho_3\rho_4},$$

defines an integral of motion of $q\mathrm{P}_{\mathrm{VI}}$.

Theorem (Joshi and PR (2022))

The six integrals of motion,

$$\eta = (\eta_{12}, \eta_{13}, \eta_{14}, \eta_{23}, \eta_{24}, \eta_{34}),$$

lie on an explicit affine algebraic surface $\mathcal{F}_q(\Theta, t_0)$, see next slide.
The induced mapping

$$\mathrm{RH}_q : \{\textit{Solutions of } qP_{VI}(\Theta, t_0)\} \to \mathcal{F}_q(\Theta, t_0), (f, g) \mapsto \eta,$$

is a one-to-one correspondence, for generic values of the parameters.

In fact, RH_q is a **diffeomorphism** when identifying the solution space of $q\mathrm{P}_{\mathrm{VI}}(\Theta, t_0)$ with the initial value space at any point [PR 2023].

Affine algebraic surface $\mathcal{F}_q(\Theta, t_0)$

The algebraic surface $\mathcal{F}_q(\Theta, t_0)$ is defined in $\{\eta \in \mathbb{C}^6\}$, by the equations

$$\begin{aligned}
&\eta_{12} + \eta_{13} + \eta_{14} + \eta_{23} + \eta_{24} + \eta_{34} = 0,\\
&a_{12}\eta_{12} + a_{13}\eta_{13} + a_{14}\eta_{14} + a_{23}\eta_{23} + a_{24}\eta_{24} + a_{34}\eta_{34} + a_\infty = 0,\\
&\eta_{13}\eta_{24} - b_1\eta_{12}\eta_{34} = 0,\\
&\eta_{14}\eta_{23} - b_2\eta_{12}\eta_{34} = 0,
\end{aligned}$$

where

$$a_{12} = \prod_{\epsilon=\pm 1} \frac{\theta_q\left(q^{+\theta_\infty} t_0\right)}{\theta_q\left(q^{\epsilon\theta_0+\theta_\infty} t_0\right)}, \qquad a_{34} = \prod_{\epsilon=\pm 1} \frac{\theta_q\left(q^{-\theta_\infty} t_0\right)}{\theta_q\left(q^{\epsilon\theta_0-\theta_\infty} t_0\right)},$$

$$a_{13} = \prod_{\epsilon=\pm 1} \frac{\vartheta_q\left(\theta_t + \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\,\theta_0 + \theta_t + \theta_1 + \theta_\infty\right)}, \quad a_{24} = \prod_{\epsilon=\pm 1} \frac{\vartheta_q\left(-\theta_t - \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\,\theta_0 - \theta_t - \theta_1 + \theta_\infty\right)},$$

$$a_{14} = \prod_{\epsilon=\pm 1} \frac{\vartheta_q\left(\theta_t - \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\,\theta_0 + \theta_t - \theta_1 + \theta_\infty\right)}, \quad a_{23} = \prod_{\epsilon=\pm 1} \frac{\vartheta_q\left(-\theta_t + \theta_1 + \theta_\infty\right)}{\vartheta_q\left(\epsilon\,\theta_0 - \theta_t + \theta_1 + \theta_\infty\right)},$$

and similar expressions for a_∞, b_1, b_2, where $\vartheta_q(x) = \theta_q(q^x)$.

Segre surfaces

The algebraic surface $\mathcal{F}_q(\Theta, t_0)$ is isomorphic to an affine **Segre surface**.

A Segre surface is by definition the intersection of two quadrics in $\mathbb{CP}^4$,

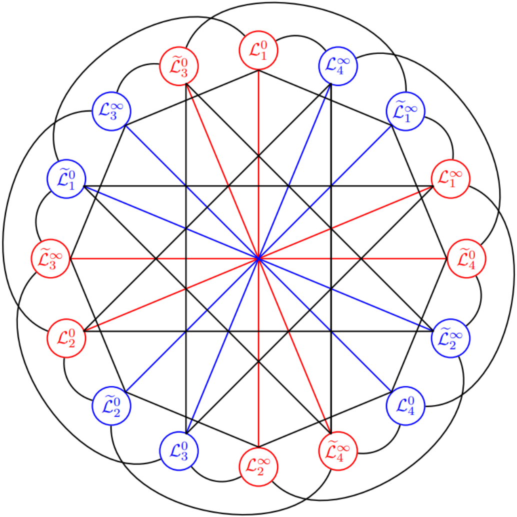
$$\{\eta \in \mathbb{CP}^4 : P(\eta) = 0\} \cap \{\eta \in \mathbb{CP}^4 : Q(\eta) = 0\},$$

where P and Q quadratic polynomials.
They were introduced and studied by Corrado Segre (1884).

What can geometry tell us?

Theorem (Corrado Segre (1884))

A generic Segre surface contains exactly **16 lines***.*



- vertices : lines
- edges : intersection points

Generic Asymptotics

Theorem (PR 2023)

Take a generic $\eta \in \mathcal{F}_q(\Theta, t_0)$, then the corresponding solution (f, g) of $qP_{VI}(\Theta, t_0)$ admits simultaneous complete asymptotic expansions,

$$f(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} F_{n,k} r_{0t}^{k} (-t)^{n+2k\sigma_{0t}},$$

$$g(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} G_{n,k} r_{0t}^{k} (-t)^{n+2k\sigma_{0t}},$$

absolutely convergent for small enough $t \in q^{\mathbb{Z}} t_0$, and

$$\frac{f(t)}{t} = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \dot{F}_{n,k} r_{01}^{k} (-t)^{-(n+2k\sigma_{01})},$$

$$\frac{1}{g(t)} = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \dot{G}_{n,k} r_{01}^{k} (-t)^{-(n+2k\sigma_{01})},$$

absolutely convergent for large enough $t \in q^{\mathbb{Z}} t_0$, with integration constants $\{\sigma_{0t}, r_{0t}\}$ and $\{\sigma_{01}, r_{01}\}$ as explicit functions of η.

Some explicit formulas

The exponents are defined through

$$\frac{\vartheta_q(\sigma_{0t}-\theta_1+\theta_\infty)\vartheta_q(\sigma_{0t}+\theta_1-\theta_\infty)}{\vartheta_q(\sigma_{0t}+\theta_1+\theta_\infty)\vartheta_q(\sigma_{0t}-\theta_1-\theta_\infty)}=\frac{T_{14}\eta_{13}}{T_{13}\eta_{14}},$$

$$\frac{\vartheta_q(\sigma_{01}-\theta_t+\theta_\infty)\vartheta_q(\sigma_{01}+\theta_t-\theta_\infty)}{\vartheta_q(\sigma_{01}+\theta_t+\theta_\infty)\vartheta_q(\sigma_{01}-\theta_t-\theta_\infty)}=\frac{T_{23}\eta_{13}}{T_{13}\eta_{23}},$$

$$0<\Re\sigma_{0t},\Re\sigma_{01}<\tfrac{1}{2},$$

and

$$r_{0t}=c_{0t}\times s_{0t},\qquad s_{0t}=M_{0t}\left(\frac{T_{34}\eta_{23}}{T_{23}\eta_{34}}\right),$$

$$r_{01}=c_{01}\times s_{01},\qquad s_{01}=M_{01}\left(\frac{T_{34}\eta_{23}}{T_{23}\eta_{34}}\right),$$

where $M_{0t}(\cdot)$ and $M_{01}(\cdot)$ are some explicit Möbius transforms and

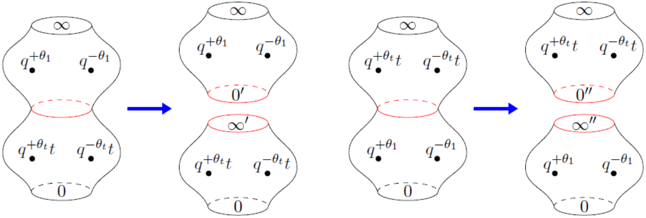
$$c_{0t}=\frac{\Gamma_q(1-2\sigma_{0t})^2}{\Gamma_q(1+2\sigma_{0t})^2}\prod_{\epsilon=\pm1}\frac{\Gamma_q(1+\theta_t+\epsilon\theta_0+\sigma_{0t})\Gamma_q(1+\theta_1+\epsilon\theta_\infty+\sigma_{0t})}{\Gamma_q(1+\theta_t+\epsilon\theta_0-\sigma_{0t})\Gamma_q(1+\theta_1+\epsilon\theta_\infty-\sigma_{0t})},$$

$$c_{01}=\frac{\Gamma_q(1-2\sigma_{01})^2}{\Gamma_q(1+2\sigma_{01})^2}\prod_{\epsilon=\pm1}\frac{\Gamma_q(1+\theta_1+\epsilon\theta_0+\sigma_{01})\Gamma_q(1+\theta_t+\epsilon\theta_\infty+\sigma_{01})}{\Gamma_q(1+\theta_1+\epsilon\theta_0-\sigma_{01})\Gamma_q(1+\theta_t+\epsilon\theta_\infty-\sigma_{01})}.$$

A short history of asymptotic studies

- Mano (2010): generic leading order asymptotics near $t = 0$ and $t = \infty$ an implicit relation between them.
- Jimbo, Nagoya and Sakai (2017): conjectural complete (and fully explicit) asymptotic expansion near $t = 0$ of the generic $q\mathrm{P}_{\mathrm{VI}}$ tau-function.
- PR (2023): complete asymptotic expansions near $t = 0$ and $t = \infty$ with explicit nonlinear connection formulas.

Mano-decompositions



(A) Decomposition I (B) Decomposition II

A generic connection matrix admits Mano-decompositions

$$z^m C(z) = C_I^i(z/t_m)(-t_m)^{\sigma_{0t}\sigma_3} C_I^e(z), \qquad |C_I^e(z)| = \theta_q(z/q^{+\theta_1}, z/q^{-\theta_1}),$$
$$= C_{II}^i(z)\,(-t_m)^{-\sigma_{01}\sigma_3} C_{II}^e(z/t_m), \quad |C_{II}^i(z)| = \theta_q(z/q^{+\theta_1}, z/q^{-\theta_1}),$$

where $t_m = q^m t_0$ and components are connection matrices of Heine hypergeometric systems.

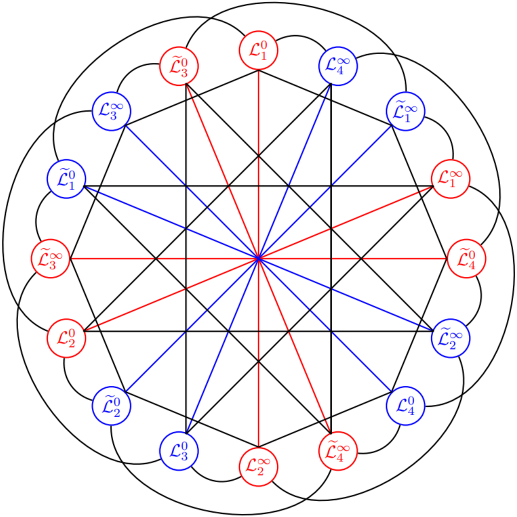
Such decompositions were first observed in Mano's asymptotic study (2010) of $q\mathrm{P}_{\mathrm{VI}}$. Proven in general by Ohyama, Ramis and Sauloy (2020).

Lines correspond to reducible factors in Mano-decompositions.

What can geometry tell us?

Theorem (Corrado Segre (1884))

A generic Segre surface contains exactly **16 lines**.



- vertices : lines
- edges : intersection points

Truncation on lines

On the blue lines the generic asymptotics near $t = 0$ **truncate**.
For example, on the line $\widetilde{\mathcal{L}}_2^\infty$, we have $\sigma_{0t} = \theta_t - \theta_0$, and

$$f(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{0} F_{n,k} r_{0t}^k (-t)^{n+2k(\theta_t-\theta_0)},$$

$$g(t) = \sum_{n=1}^{\infty} \sum_{k=-n}^{0} G_{n,k} r_{0t}^k (-t)^{n+2k(\theta_t-\theta_0)},$$

if $\Re(\theta_t - \theta_0) < \frac{1}{2}$.
On the **intersection point** of blue lines $\widetilde{\mathcal{L}}_2^\infty$ and $\widetilde{\mathcal{L}}_1^0$, we have $r_{0t} = 0$ and the generic asymptotics are **doubly truncated**,

$$f(t) = \sum_{n=1}^{\infty} F_{n,0}(-t)^n, \qquad F_{1,0} = \frac{q^{\theta_0} - q^{-\theta_0}}{q^{\theta_t-\theta_0} - q^{\theta_0-\theta_t}},$$

$$g(t) = \sum_{n=1}^{\infty} G_{n,0}(-t)^n, \qquad G_{1,0} = \frac{q^{\theta_t} - q^{-\theta_t}}{q^{\theta_0-\theta_t} - q^{\theta_t-\theta_0}}.$$

The latter power series solutions should be called Kaneko-Ohyama solutions (2013,2015).

black intersection points

Let (f, g) be the solution corresponding to the intersection point

$$\{\eta_*(t)\} = \widetilde{\mathcal{L}}_1^0 \cap \widetilde{\mathcal{L}}_3^\infty,$$

and assume $\Re(\theta_0 - \theta_t), \Re(-\theta_0 - \theta_1) < \frac{1}{2}$, then $f(t)$ admits simultaneous uniformly convergent asymptotic expansions

$$f(t) = \frac{q^{\theta_0} - q^{-\theta_0}}{q^{\theta_0-\theta_t} - q^{\theta_t-\theta_0}}\, t + tE_0(t) + \sum_{n=2}^{\infty}\sum_{k=0}^{n} f_{n,k} t^n E_0(t)^k \qquad (t \to 0),$$

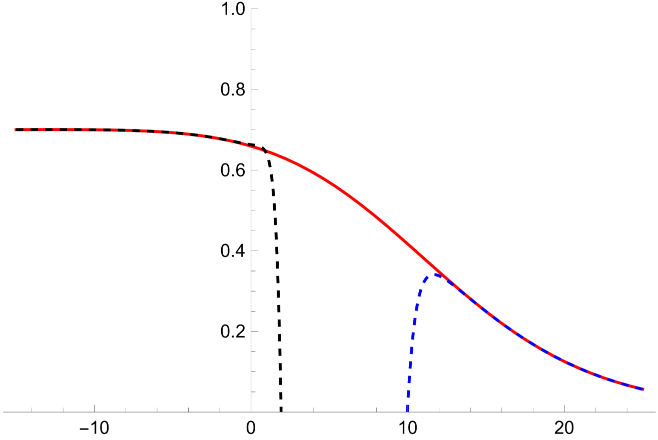
$$f(t) = \frac{q^{\theta_0} - q^{-\theta_0}}{q^{\theta_0+\theta_1} - q^{-\theta_0-\theta_1}} + E_\infty(t) + t\sum_{n=2}^{\infty}\sum_{k=0}^{n} \dot{f}_{n,k} t^{-n} E_\infty(t)^k \quad (t \to \infty),$$

on compact sets $K \subseteq \mathbb{CP}^1 \setminus q^{\mathbb{Z}-2\theta_0+\theta_t-\theta_1}$, with $qK = K$, where

$$E_0(t) = c_0 \frac{\theta_q(q^{-\theta_t-\theta_1} t)}{\theta_q(q^{-2\theta_0+\theta_t-\theta_1} t)}, \qquad E_\infty(t) = c_\infty \frac{\theta_q(q^{-\theta_t-\theta_1} t^{-1})}{\theta_q(q^{+2\theta_0-\theta_t+\theta_1} t^{-1})},$$

for some explicit constant factors c_0, c_∞.

Plot of f on negative real line



Plot of $f(-q^r)$ in red with $r \in (-15, 25)$ and parameter values

$$\theta_0 = \frac{1}{3}, \quad \theta_t = \frac{1}{5}, \quad \theta_1 = \frac{1}{7}, \quad \theta_\infty = \frac{1}{11}, \quad q = \exp(-\tfrac{1}{4}).$$

In dashed black and blue the series expansions around $t = -\infty$ and $t = 0$ respectively.

Continuum limit

As $q \uparrow 1$, the algebraic surface $\mathcal{F}_q(\Theta, t_0)$ becomes

$$\eta_{12} + \eta_{13} + \eta_{14} + \eta_{23} + \eta_{24} + \eta_{34} = 0,$$
$$\eta_{12} + \eta_{13} + a_{14}\eta_{14} + a_{23}\eta_{23} + a_{24}\eta_{24} + a_{34}\eta_{34} - 1 = 0,$$
$$\eta_{13}\eta_{24} - b_1\eta_{12}\eta_{34} = 0,$$
$$\eta_{14}\eta_{23} - b_2\eta_{12}\eta_{34} = 0,$$

where

$$a_{13} = \prod_{\epsilon=\pm1} \frac{\sin(\pi(\vartheta_t + \vartheta_1 + \vartheta_\infty))}{\sin(\pi(\epsilon\vartheta_0 + \vartheta_t + \vartheta_1 + \vartheta_\infty))}, \quad a_{24} = \prod_{\epsilon=\pm1} \frac{\sin(\pi(-\vartheta_t - \vartheta_1 + \vartheta_\infty))}{\sin(\pi(\epsilon\vartheta_0 - \vartheta_t - \vartheta_1 + \vartheta_\infty))},$$
$$a_{14} = \prod_{\epsilon=\pm1} \frac{\sin(\pi(\vartheta_t - \vartheta_1 + \vartheta_\infty))}{\sin(\pi(\epsilon\vartheta_0 + \vartheta_t - \vartheta_1 + \vartheta_\infty))}, \quad a_{23} = \prod_{\epsilon=\pm1} \frac{\sin(\pi(-\vartheta_t + \vartheta_1 + \vartheta_\infty))}{\sin(\pi(\epsilon\vartheta_0 - \vartheta_t + \vartheta_1 + \vartheta_\infty))},$$

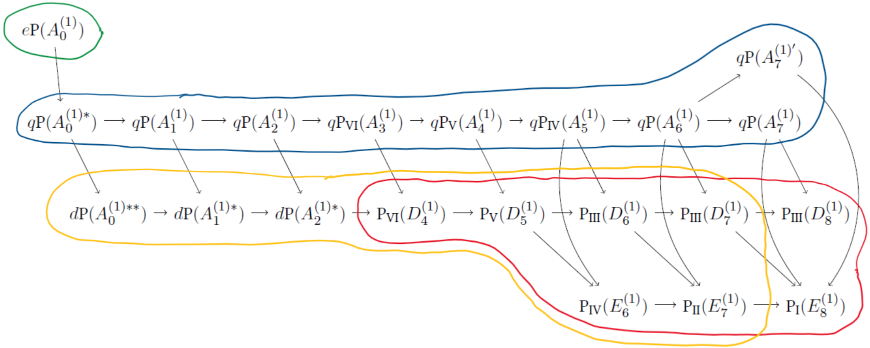
and similar expressions for b_1, b_2.

Theorem (Joshi, Mazzocco, PR (to appear))

The limiting Segre surface is isomorphic to the Jimbo-Fricke cubic surface with one line at infinity blown down.

Outlook

Sakai (2001) classified all Painlevé equations, differential and discrete, in terms of their initial value spaces.



Can these methods be extended to the other discrete Painlevé equations? What are the algebraic surfaces on the right-hand sides of the Riemann-Hilbert correspondence for them?