Singularities of Painlevé functions, Heun equations and generalised Hermite polynomials

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Dualities and Symmetries in Integrable Systems

References

Based on joint work with Davide Masoero (University of Lisbon):

- D. Masoero and PR - *Poles of Painleve IV Rationals and their Distribution*, SIGMA, 2018.
- D. Masoero and PR - *Roots of the Generalised Hermite Polynomials when both Parameters are Large*, Nonlinearity, 2021.
- D. Masoero and PR - *Large Parameter Asymptotics of Painlevé IV Functions*, in preparation.

Generalised Hermite polynomials

This talk is on Wronskians of consecutive Hermite polynomials

$$H_{m,n} = \mathcal{W}(h_m, h_{m+1}, \ldots, h_{m+n-1}) \quad (m, n \in \mathbb{N}),$$

also known as **generalised Hermite polynomials**, where h_k denotes the k-th Hermite polynomial.

They appear in various contexts related to integrability:

- Monodromy free operators (Oblomkov 1999, Veselov 2000)
- Nonlinear wave equations (Clarkson and Thomas 2009)
- Random matrix theory (Forrester and Witte 2001, Chen and Feigin 2006)
- 2-D Coulomb gas in a quadratic potential (Marikhin 2001)
- Exceptional orthogonal polynomials (Kuijlaars and Milson 2014)

⋮

- **Monodromy preserving deformations of certain linear ODEs**

Motivation

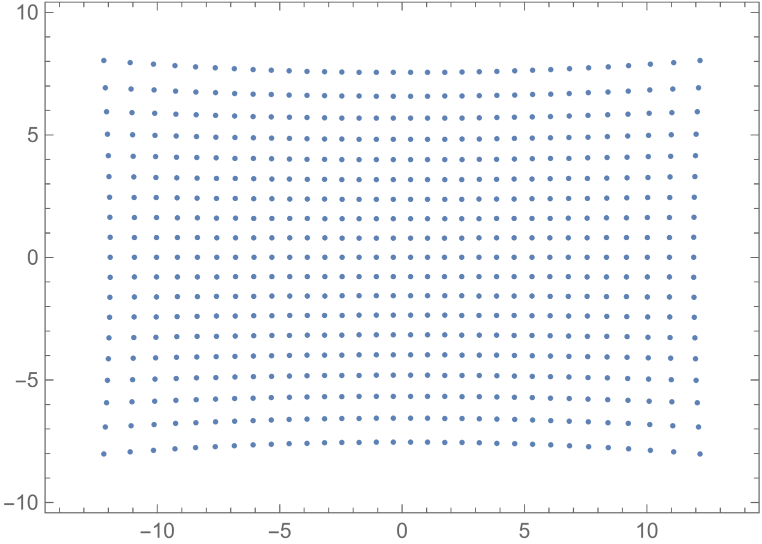


Figure: Roots of $H_{m,n}$ in $\mathbb{C}$, with $m = 32$ and $n = 19$.

Motivation

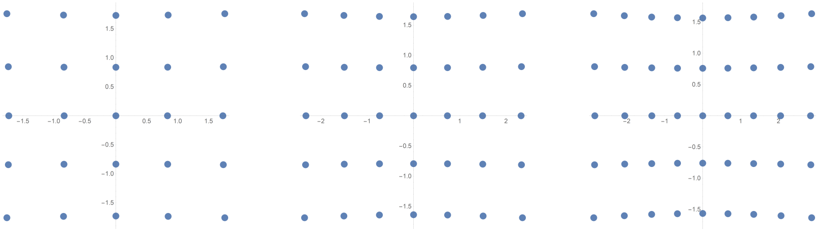


Figure: Roots of $H_{m,n}$, with $n = 5$ and $m = 5, 7, 9$

Problem (Clarkson, 2003)

The roots seem to lie on a deformed rectangular lattice. Is there an analytic description of the roots explaining this regularity?

We will consider this problem asymptotically as $m + n \to \infty$.

Rational solutions of P_{IV}

Theorem (Noumi and Yamada 1999)

For $m, n \in \mathbb{N}$,

$$u_{m,n} = \frac{H'_{m+1,n}}{H_{m+1,n}} - \frac{H'_{m,n}}{H_{m,n}}, \quad \theta_0 = \tfrac{1}{2}n, \quad \theta_\infty = m + \tfrac{1}{2}n + 1,$$

defines a rational solution of the fourth Painlevé equation

$$P_{IV}: \quad u_{tt} = \frac{1}{2u}u_t^2 + \tfrac{3}{2}u^3 + 4tu^2 + 2(t^2 + 1 - 2\theta_\infty)u - \frac{8\theta_0^2}{u}.$$

Note: poles of rational solutions coincide with roots of generalised Hermite polynomials.

Isomononodomic deformations

- Almost since the discovery of the Painlevé equations, it is known that they govern **monodromy preserving deformations** of certain linear ODEs in the complex domain (R. Fuchs 1905, Garnier 1912, Jimbo et al. 1981).
- In scalar form, the linear ODEs can be taken as (confluent) **Heun equations** with an additional apparent singularity.
- Example: R. Fuchs constructed an ODE with four regular singular points at $z = 0, t, 1, \infty$, and an apparent singularity at $z = u$,

$$Y_{zz} = V\,Y,$$

$$V = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_t^2 - \frac{1}{4}}{(z-t)^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\frac{3}{4}}{(z-u)^2} + \frac{A}{z} + \frac{B}{z-t} + \frac{C}{z-1} + \frac{p}{z-u},$$

  whose monodromy is independent of t, where $u = u(t)$ necessarily satisfies the sixth Painlevé equation!
- At movable singularities of the Painlevé equation, the linear ODE reduces to the corresponding Heun equation.

Movable singularities and Heun equations

- Painlevé VI:

$$u_{tt} = \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\frac{u_t^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u_t$$
$$+\frac{u(u-1)(u-t)}{2t^2(t-1)^2}\left((2\theta_\infty - 1)^2 - \frac{4\theta_0^2 t}{u^2} + \frac{4\theta_1^2(t-1)}{(u-1)^2} + \frac{(1-4\theta_t^2)t(t-1)}{(u-t)^2}\right).$$

- $t = a$ is a **movable singularity** of Painlevé VI if $u(a) \in \{0, a, 1, \infty\}$.
- Near a zero $t = a$,

$$u(t) = \frac{\pm 2\theta_0}{1-a}(t-a) + b(t-a)^2 + \mathcal{O}(t-a)^3, \quad (t \to a),$$

and Fuchs' ODE reduces to the **Heun equation**,

$$Y_{zz} = V\,Y,$$
$$V = \frac{(\theta_0 \pm \frac{1}{2})^2 - \frac{1}{4}}{z^2} + \frac{\theta_t^2 - \frac{1}{4}}{(z-a)^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{A(b)}{z} + \frac{B(b)}{z-a} + \frac{C(b)}{z-1}.$$

Painlevé/Heun Duality

Movable singularities of P-eqns $\leftrightarrow$ Heun eqns $Y_{zz} = V(z; a, b)Y$

P-eqns	Heun eqns	some potentials
P_{VI}	Classical Heun	-
P_{V}	Confluent Heun	$-\frac{\theta_0^2-\frac{1}{4}}{z^2} + \frac{\theta_t^2-\frac{1}{4}}{(z-a)^2} + \frac{\theta_\infty}{z} - \frac{b}{z(z-a)} + \frac{1}{4}$
P_{IV}	Biconfluent Heun	$(z+a)^2 + 2(1-\theta_\infty) - \frac{b}{z} + \frac{\theta_0^2-\frac{1}{4}}{z^2}$
$P_{\mathrm{III}}^{D_6}$	Doubly confluent He	$\frac{1}{4} + \frac{\theta_\infty}{z} - \frac{b}{z^2} + \frac{\theta_0 a}{z^3} + \frac{a^2}{4z^4}$
P_{II}	Triconfluent Heun	$(z^2+a)^2 + 2\theta_\infty z + b$
P_{I}	cubic oscillator	$4z^3 - 2az + b$

See e.g. Lisovyy and Naidiuk (2018).

Classification

Theorem (D. Masoero and PR, 2018)

For $m, n \in \mathbb{N}$, the point $a \in \mathbb{C}$ is a root of $H_{m,n}$ if and only if there exists an (a fortiori unique) $b \in \mathbb{C}$ such that the biconfluent Heun equation

$$\psi''(z) = \left(z^2 + 2a\,z + a^2 - (2m+n) - \frac{b}{z} + \frac{n^2-1}{4z^2}\right)\psi(z), \qquad (1)$$

satisfies the following conditions:

- **Apparent Singularity Condition.** *The monodromy around Fuchsian singularity $z = 0$ is trivial, namely*

$$\psi(e^{2\pi i}z) = (-1)^{n+1}\psi(z)\,, \quad \forall\psi \text{ solution of } (1).$$

- **Quantisation Condition.** *There exists a non-zero solution of (1) which solves the following boundary value problem*

$$\lim_{z\to+\infty}\psi(z) = \lim_{z\to 0^+}\psi(z) = 0\,.$$

Nevanlinna-Elfving function characterisation

$$\psi''(z) = (z^2 + 2a\,z + a^2 - (2m+n) - \frac{b}{z} + \frac{n^2-1}{4z^2})\psi(z).$$

Any ratio of solutions $f = \psi/\phi : \mathbb{C} \to \mathbb{CP}^1$ defines a meromorphic function, that satisfies

- f has a critical point of order $n-1$ at $z = 0$,
- In each of the four Stokes sectors,

$$\Sigma_k = \{z \in \mathbb{Z} : |\arg z - \frac{k\pi}{2}| < \frac{\pi}{2}\}, \quad k \in \mathbb{Z}_4,$$

  f has a logarithmic direct transcendental singularity,

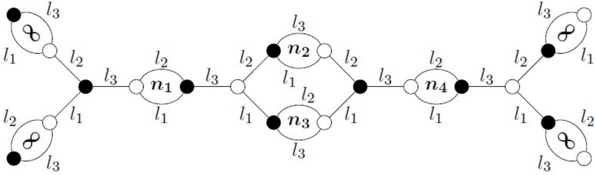
$$f(z) \to w_k, \quad (z \to \infty, z \in \Sigma_k),$$

  exponentially fast, for some $w_k \in \mathbb{CP}^1$.
- The critical value $f(0) = w_*$, at $z = 0$, and the critical values at $z = \pm\infty$ coincide,

$$w_* = w_0 = w_2.$$

Nevanlinna-Elfving function characterisation, continued

- So f is a branched covering of $\mathbb{CP}^1$ ramified over only three points (non-compact Belyi function).
- It sits in a larger class of meromorphic functions introduced and studied by Nevanlinna (1932), and his student Elfving (1934).
- Its dessin d'enfant is given by



for some $(n_1, n_2, n_3, n_4) \in \mathbb{N}^4$ with $n_1 + n_4 = m - 1$, $n_2 + n_3 = n - 1$.

Corollary

$H_{m,n}$ has m real roots when n is odd and none when n is even.

Proof. Count the number of dessin d'enfants invariant under horizontal reflection. □

Rescaling

Setting

$$E = 2m + n, \quad \boldsymbol{\alpha} = E^{-\frac{1}{2}} \boldsymbol{a}, \quad \boldsymbol{\beta} = E^{-\frac{3}{2}} \boldsymbol{b}, \quad \nu = \frac{n}{E},$$

we have:

$t = \boldsymbol{\alpha}$ is a root of $H_{m,n}(E^{\frac{1}{2}} t)$ if and only if $\exists \boldsymbol{\beta}$ such that

$$\psi''(z) = \left(E^2 V(z; \boldsymbol{\alpha}, \boldsymbol{\beta}, \nu) - \frac{1}{4z^2} \right) \psi(z),$$

$$V = z^2 + 2\boldsymbol{\alpha} z + \boldsymbol{\alpha}^2 - 1 - \boldsymbol{\beta} z^{-1} + \frac{\nu^2}{4} z^{-2},$$

satisfies **apparent singularity** and **quantisation condition**.

Next step: Apply complex WKB approach as $E \to \infty$ to approximate ratios of critical values of ratios of solutions (**dankjewel Nevanlinna!**) and solve apparent singularity and quantisation conditions.

Complex WKB Approach

As $E \to \infty$ solutions of anharmonic oscillator are (locally) well-approximated by WKB functions

$$\psi = V^{-\frac{1}{4}} e^{\pm E \int^z \sqrt{V(\mu)} d\mu},$$

$$V = z^2 + 2\alpha z + \alpha^2 - 1 - \beta z^{-1} + \frac{\nu^2}{4} z^{-2}.$$

Ratios of critical values of ratios of solutions are approximated by exponentials of complete elliptic integrals of the form

$$\exp E \oint_\gamma \frac{y}{z} dz,$$

where γ a cycle on the elliptic curve

$$y^2 = z^2 V(z) = z^4 + 2\alpha z^3 + (\alpha^2 - 1) z^2 - \beta z + \frac{\nu^2}{4}.$$

Boutroux curve

As $E \to \infty$, the quantisation and apparent singularity conditions become asymptotically equivalent to

$$\exp E \oint_{\gamma_1} \frac{y}{z} dz = 1, \quad \exp E \oint_{\gamma_2} \frac{y}{z} dz = 1,$$

for some homologically independent cycles $\gamma_{1,2}$ on the elliptic curve, dependent on the underlying Stokes topology of the potential.

$\implies$ A necessary condition to asymptotically solve the quantisation and apparent singularity conditions is that the elliptic curve is a **Boutroux curve**:

$$\mathfrak{R} \oint_{\gamma} \frac{y}{z} dz = 0,$$

for any cycle γ.

Imposing that the elliptic curve is a Boutroux curve splits the α-plane up into different chambers, each characterised by topologically distinct **Stokes complexes** of the potential.

Stokes Geometry

Consider potential

$$V(z) = z^2 + 2\boldsymbol{\alpha}\, z + \boldsymbol{\alpha}^2 - 1 - \boldsymbol{\beta}\, z^{-1} + \frac{\nu^2}{4} z^{-2}.$$

- **Turning points** are the zeros of $V(z)$.
- **Stokes lines** are level sets $\Re \int_{z^*}^{z} \sqrt{V(\mu)} d\mu = 0$ in $\mathbb{CP}^1$, where z^* any turning point.
- **Stokes complex** $\mathcal{S} = \mathcal{S}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \subseteq \mathbb{CP}^1$ of $V(z)$ is the union of all its Stokes lines and zeros (decorated with some points at infinity).

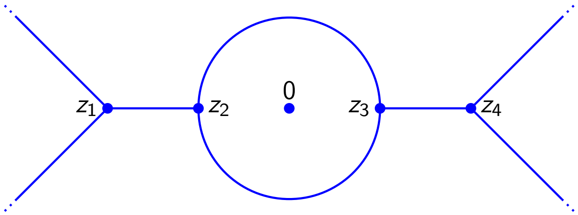


Figure: Topological representation of Stokes complex $\mathcal{S}(\alpha, \beta)$ with $(\alpha, \beta) = (0, 0)$, where $z_{1,2,3,4}$ are the zeros of $V = z^2 - 1 + \frac{\nu^2}{4} z^{-2}$.

The central chamber

Definition (The central chamber)

Let R be region where Stokes complex is isomorphic to Stokes complex at $(0,0)$,

$$R = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{C}^2 : \mathcal{S}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cong \mathcal{S}(0,0)\}.$$

Define the central chamber $\mathcal{C} = \mathcal{C}(\nu)$ as the projection of $\overline{R}$ onto the $\boldsymbol{\alpha}$-plane.

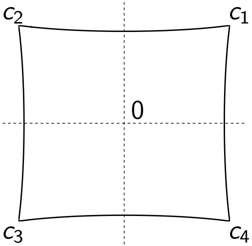
The central chamber, corners

Theorem (The central chamber, part 1, PR and Masoero 2021)

The central chamber $\mathcal{C}$ is a compact quadrilateral domain whose boundary is a Jordan curve composed of four analytic pieces (edges), meeting at four corners $c_{1,2,3,4}$, as in figure.
The corner c_k is the unique solution of

$$\alpha^8 - 6(3\nu^2 + 1)\alpha^4 + 8(1 - 9\nu^2)\alpha^2 - 3(9\nu^4 + 6\nu^2 + 1) = 0$$

in k-th quadrant of complex α-plane. (Remaining four roots are purely real or imaginary)



The central chamber, boundary parametrisation

Theorem (The central chamber, part 2)

Cut α-plane along diagonals $[c_1, c_3]$ and $[c_2, c_4]$. Then

$$\psi(\alpha) = \tfrac{1}{2}\Re\left[\alpha y + \tfrac{1}{2}(1-\nu)\log(p_1) - \log(p_2) + \nu\log(x^{-2}p_3)\right],$$

$$p_1 = 1 - 2x\alpha - 2x^2, \quad p_2 = 2x + \alpha + y, \quad p_3 = x(\alpha^2 + 5x\alpha + 4x^2 - 1) + \frac{1}{2}\nu y,$$

is a **univalued harmonic function** *on this cut plane.*

Here $x = x(\alpha)$ and $y = y(\alpha)$ are the unique algebraic functions which solve

$$3x^4 + 4\alpha x^3 + (\alpha^2 - 1)x^2 - \frac{\nu^2}{4} = 0, \qquad x(\alpha) \sim \frac{\nu}{2}\alpha^{-1} \quad (\alpha \to \infty),$$

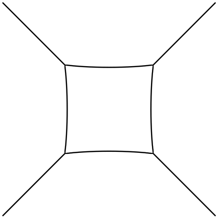
$$y^2 = \alpha^2 + 6x\alpha + 6x^3 - 1, \qquad y(\alpha) \sim \alpha \quad (\alpha \to \infty),$$

on the same cut plane.

The central chamber, dense filling

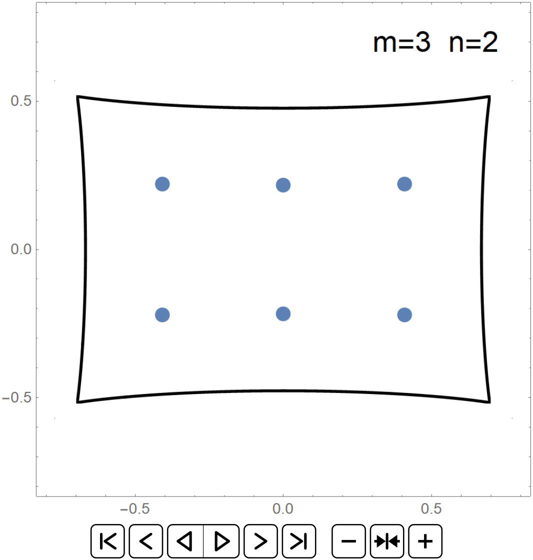
Theorem (The central chamber, part 3)

The level set $\{\psi(\alpha) = 0\}$ *consists of the* **boundary** *of the central chamber* $\partial\mathcal{C}$ *plus four additional lines which emanate from corners and go to infinity, see figure.*
As $E \to \infty$*, roots of* $H_{m,n}(E^{\frac{1}{2}}t)$ **densely fill** *up the central chamber.*



Buckingham (2018) obtained different parametrisation of central chamber via Riemann-Hilbert approach to certain orthogonal polynomials and proved:
asymptotically there are **no roots outside elliptic region**.

Dense filling of central chamber, $\nu = \frac{1}{4}$, $\frac{m}{n} = \frac{3}{2}$



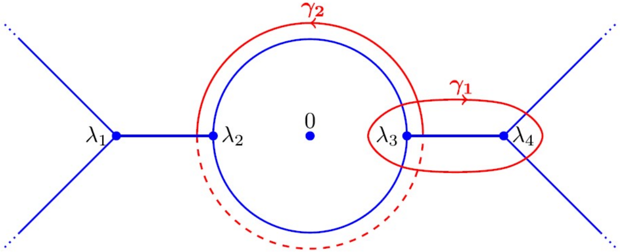
Two Complete Elliptic Integrals

To describe asymptotic distribution of zeros within the central chamber, we require the following two complete elliptic integrals,

$$s_1(\alpha, \beta) = \int_{\gamma_1} \frac{y}{z} dz + \frac{i\pi(1-\nu)}{2},$$
$$s_2(\alpha, \beta) = \int_{\gamma_2} \frac{y}{z} dz.$$

Here γ_1 lies in sheet where $y \sim \frac{\nu}{2z}$ as $z \to 0$ of elliptic curve

$$y^2 = z^2 V(z) = z^4 + 2\alpha z^3 + (\alpha^2 - 1) z^2 - \beta z + \frac{\nu^2}{4}.$$



Asymptotic distribution within central chamber

Informally (see paper for rigorous results), as E grows large, the **quantisation** and **apparent singularity conditions** are asymptotically equivalent to respectively

$$s_1(\alpha,\beta) = i\frac{\pi j}{E}, \quad j \in \{-m+1, -m+3, \dots, +m-1\},$$
$$s_2(\alpha,\beta) = i\frac{\pi k}{E}, \quad k \in \{-n+1, -n+3, \dots, +n-1\}.$$

We may eliminate β by imposing $\Re s_{1,2}(\alpha,\beta) = 0$.
Then equations

$$\Im s_1(\alpha) = \frac{\pi j}{E}, \quad j \in \{-m+1, -m+3, \dots, +m-1\}$$

define m 'vertical' grid lines within $\mathcal{C}$.
Similarly equations

$$\Im s_2(\alpha) = \frac{\pi k}{E}, \quad k \in \{-n+1, -n+3, \dots, +n-1\}$$

define n 'horizontal' grid lines within $\mathcal{C}$.

Deformed rectangular lattice

Example $(m, n) = (4, 3)$:

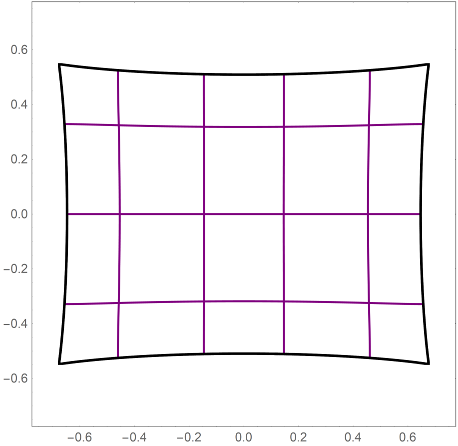


Figure: **deformed rectangular lattice** within central chamber $\mathcal{C}$

Asymptotic distribution within central chamber

Theorem (Informally, see paper for rigorous statement)

In the large E limit, the bulk of the (rescaled) roots organise themselves within the central chamber along the ***vertices*** *of the* ***deformed rectangular lattice****.*

Example $(m, n) = (4, 3)$:

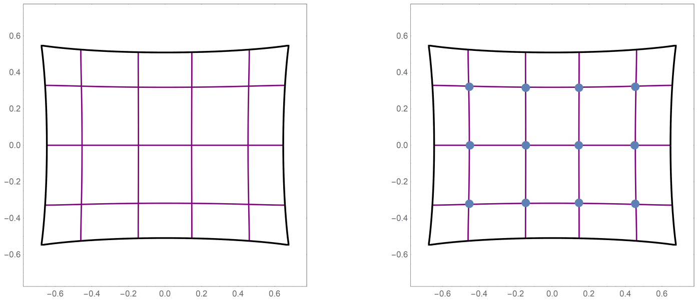


Figure: In both figures **deformed rectangular lattice** with on the right true locations of **roots** **superimposed**.

Asymptotic Distribution

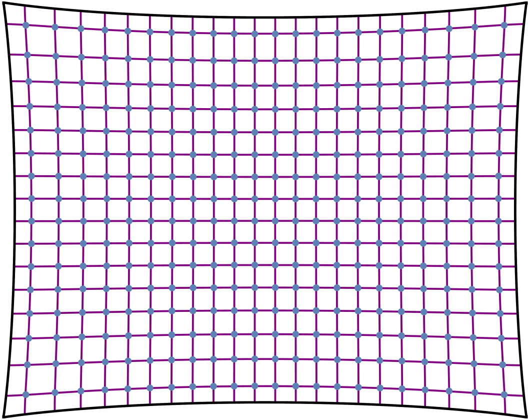


Figure: Asymptotic prediction are vertices of purple lattice, true location roots $H_{m,n}(z)$ in blue, with $(m,n) = (22,16)$.

Comments on literature

- Buckingham (2018) proved that asymptotically there are no zeros outside the elliptic region. Method: Riemann-Hilbert approach to associated orthogonal polynomials.
- PR and Masoero (2018,2021), proved that zeros densely fill up elliptic region, organising themselves along a deformed rectangular lattice.
- Buckingham and Miller (2022) computed asymptotic behaviour of rational solutions on the in and outside of elliptic region, by Riemann-Hilbert approach to the linear problem.

$E \to \infty,\ n = \mathcal{O}(1).$

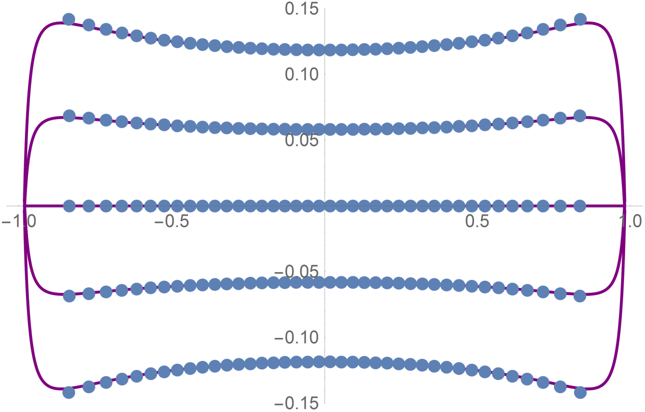


Figure: $(m, n) = (40, 5)$

Letting $E \to \infty$ with n fixed, the zeros condensate on n curves, with their real parts distributed following Wigner's semi-circle law.
See Felder, Hemery, Veselov (2012) for similar results/conjectures for more general Wronskians of Hermite polynomials.