Wronskians of Hermite polynomials, anharmonic oscillators and Painlevé IV

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Based on joint works with Davide Masoero:
Poles of Painlevé IV Rationals and their Distribution, SIGMA, 2018;
Roots of generalised Hermite polynomials when both parameters are large, ArXiv, 2019.

Preamble

- This talk is on the **generalised Hermite polynomials**

$$H_{m,n} = \mathcal{W}(h_m, h_{m+1}, \ldots, h_{m+n-1}) \quad (m, n \in \mathbb{N}),$$

  where h_k denotes k-th Hermite polynomial.
- These polynomials generate **rational solutions** of the **fourth Painlevé equation** and appear in various applications:
  - quantum mechanics (Marquette and Quesne)
  - interesting combinatorics (Dunning et al)
  - nonlinear wave equations (Clarkson)
  - random matrix theory (Forrester and Witte, Chen and Feigin)
  - 2-d Coulomb gas in a quadratic potential (Veselov, Marikhin)

Main focus: root distributions

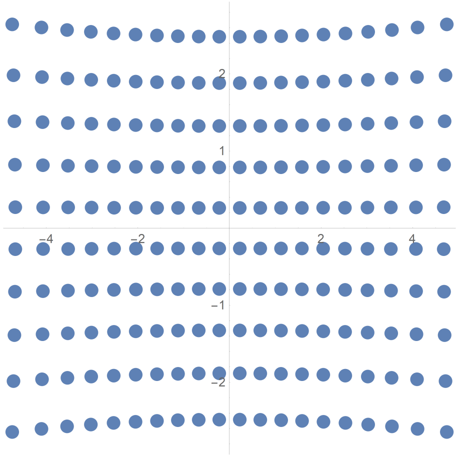


Figure: Roots of $H_{m,n}$ in complex plane with $(m, n) = (20, 10)$

Hermite Rationals (Noumi and Yamada, 1999)

For $m, n \in \mathbb{N}$,

$$\omega_{m,n} = \frac{H'_{m+1,n}}{H_{m+1,n}} - \frac{H'_{m,n}}{H_{m,n}}, \quad \theta_0 = \tfrac{1}{2}n, \quad \theta_\infty = m + \tfrac{1}{2}n + 1,$$

defines a rational solution of the fourth Painlevé equation

$$P_{\mathrm{IV}}: \quad \omega_{zz} = \frac{1}{2\omega}\omega_z^2 + \tfrac{3}{2}\omega^3 + 4z\omega^2 + 2(z^2 + 1 - 2\theta_\infty)\omega - \frac{8\theta_0^2}{\omega}.$$

Note: poles of rational solutions coincide with roots of generalised Hermite polynomials!

Determining **root distributions** of generalised Hermite polynomials is equivalent to determining **pole distributions** of certain P_{IV} functions.

Pole distributions of Painlevé functions

- The problem (project) of determining pole distributions of Painlevé functions is long-standing
- Known theoretical result: non-rational P_{I}-P_{V} functions have infinitely many poles (Joshi et al, Laine et al)
- Only for a limited number of Painlevé functions very explicit results on pole distribution have been obtained:
    - Painlevé I: the tritronquée solution, Boutroux (1913), Joshi and Kitaev (2001), Costin et al (2014), Masoero (2010-2014).
    - Painlevé II: rational solutions, Buckingham and Miller (2014,2015), Bertola and Bothner (2015).
    - Painlevé II/III: real solutions, Its and Novokshenov (1986).
    - Painlevé III: rational solutions, Bothner and Miller (2018)
    - **Painlevé IV: rational solutions (Hermite)**, Buckingham (2018), Masoero and PR (2018,2019)

      ⋮

    - Painlevé VI: Picard-Hitchin solutions, Brezhnev (2010)
    - Painlevé VI: real solutions, Eremenko and Gabrielov (2017)
    - Painlevé VI: hypergeometric-type solutions, Dubrovin and Kapaev (2018)

Overview

Generalised Hermite Polynomials

Degree:

$$\deg(H_{m,n}) = m \times n$$

Symmetry:

$$H_{n,m}(z) = i^{mn} H_{m,n}(-iz)$$

Examples:

$$H_{m,1}(z) = h_m(z) \quad (m \in \mathbb{N}),$$
$$H_{2,2}(z) = z^4 + 12$$
$$H_{3,2}(z) = z^6 - 6z^4 + 36z^2 + 72$$
$$H_{3,3}(z) = z^9 + 72z^5 - 2160z$$
$$H_{4,2}(z) = z^8 - 16z^6 + 120z^4 + 720$$
$$H_{4,3}(z) = z^{12} - 12z^{10} + 180z^8 - 480z^6 - 3600z^4 - 43200z^2 + 43200$$
$$H_{4,4}(z) = z^{16} + 240z^{12} - 7200z^8 + 2016000z^4 + 6048000$$

Roots of generalised Hermite polynomials

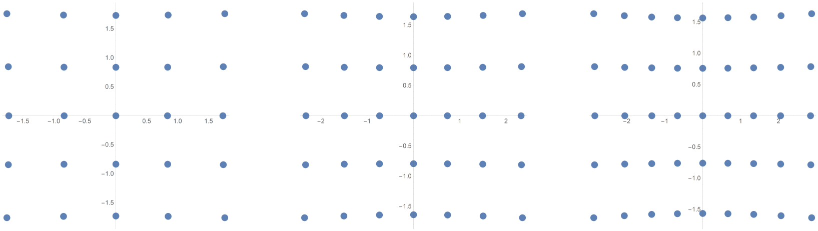


Figure: Roots of $H_{m,n}$, with $n = 5$ and $m = 5, 7, 9$

Problem (Clarkson, 2003)

The roots seem to lie on a deformed rectangular lattice. Is there an analytic description of the roots explaining this regularity?

The elliptic region

Setting

$$E = 2m + n, \quad n = E\nu,$$

and keeping $\nu > 0$ fixed, so ratio

$$\frac{m}{n} = \frac{1-\nu}{2\nu} \quad \text{fixed,}$$

the roots of

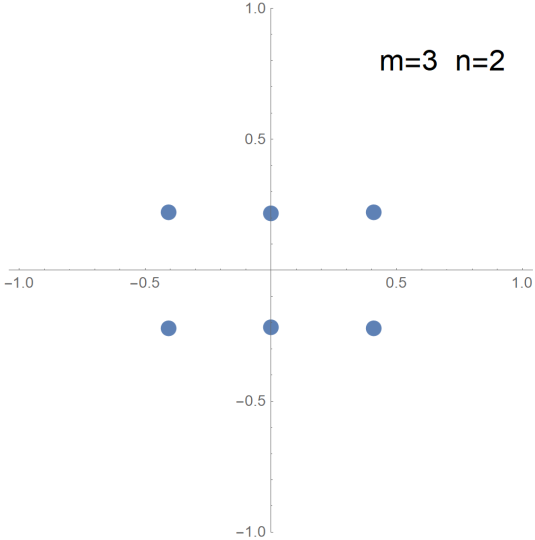
$$H_{m,n}(E^{\frac{1}{2}} z)$$

seem to condensate on compact region $K = K(\nu) \subseteq \mathbb{C}$ as $E \to \infty$.

Problem

Determine the 'elliptic region' $K = K(\nu) \subseteq \mathbb{C}$ and prove that the roots indeed densely fill this region as $E \to \infty$.

The elliptic region, $\nu = \frac{1}{4}$, $\frac{m}{n} = \frac{3}{2}$



Strategy

- **Part 1: exploit integrability of $P_{\mathbf{IV}}$**
  result: roots $z = a$ of generalised hermite polynomials $H_{m,n}(z)$ are related to anharmonic oscillators

  $$\psi''(\lambda) = (\lambda^2 + 2a\lambda + a^2 - (2m+n) - \frac{b}{\lambda} + \frac{n^2-1}{4\lambda^2})\psi(\lambda),$$

  satisfying two quantisation conditions.
- **Part 2: a complex WKB approach** to oscillators
  result: Description of elliptic region plus asymptotic distribution of roots as $m, n \to \infty$.

Roots and Anharmonic Oscillators

Integrability of Painlevé equations

- Each **Painlevé equation** $P_K(\theta)$: $\omega_{zz} = R_k(\omega, \omega_z, z; \theta)$, $K = I, \dots VI$, has an associated isomonodromic linear system

$$Y_\lambda = A_K(\lambda; \omega, \omega_z, z, \theta) Y,$$

that is, as z moves, the **monodromy data** of system remain invariant. (Jimbo and Miwa, 1981)

- Monodromy data form **complete set of first integrals** of corresponding Painlevé equation:

$$\mathcal{M} : \{\text{solutions of } P_K(\theta)\} \to \{\text{monodromy data}\} \quad \text{injective.}$$

- At **pole** of ω, the linear system $Y_\lambda = AY$ is either singular or degenerates to a (confluent) **Heun equation**.
- This allows for **poles** of solutions to be characterised in terms of **inverse monodromy problems** concerning (confluent) **Heun equations**.

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Evaluating PIV isomonodromic system at a root

Take root $z = a$ of $H_{m,n}(z)$, then

$$\omega_{m,n} = \frac{H'_{m+1,n}}{H_{m+1,n}} - \frac{H'_{m,n}}{H_{m,n}} = -\frac{1}{z-a} - a + c(z-a) + b(z-a)^2 + \mathcal{O}(z-a)^3$$

with $c = -\frac{1}{3}(a^2 + 4m + 2n + 6)$.
Jimbo-Miwa linear system $Y_\lambda = A(\lambda, z)Y$ is regular and degenerates at $z = a$ to system form of

$$\psi_{\lambda\lambda} = V(\lambda; a, b, m, n)\psi,$$

$$V = \lambda^2 + 2a\lambda + a^2 - (2m+n) - \frac{b + (2m+n+\frac{3}{2})a}{\lambda} + \frac{n^2-1}{4\lambda^2}.$$

- This is an anharmonic oscillator.
- More precisely, it's a harmonic oscillator + Fuchsian singularity.
- Known also as a **biconfluent Heun equation**.

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Characterisation Roots

Theorem (D. Masoero and PR, 2018)

For $m, n \in \mathbb{N}$, the point $a \in \mathbb{C}$ is a root of $H_{m,n}$ if and only if there exists an (a fortiori unique) $b \in \mathbb{C}$ such that the anharmonic oscillator

$$\psi''(\lambda) = (\lambda^2 + 2a\lambda + a^2 - (2m+n) - \frac{b}{\lambda} + \frac{n^2-1}{4\lambda^2})\psi(\lambda), \tag{1}$$

satisfies the following two properties:

1. **Apparent Singularity Condition.** *The monodromy around Fuchsian singularity $\lambda = 0$ is scalar. In a formula,*

$$\psi(e^{2\pi i}\lambda) = (-1)^{n+1}\psi(\lambda), \quad \forall \psi \text{ solution of } (1).$$

2. **Quantisation Condition.** *There exists a non-zero solution of (1) which solves the following boundary value problem*

$$\lim_{\lambda \to +\infty} \psi(\lambda) = \lim_{\lambda \to 0^+} \psi(\lambda) = 0 .$$

Complex WKB Approach

Rescaling

Setting

$$E = 2m + n, \quad \alpha = E^{-\frac{1}{2}} a, \quad \beta = E^{-\frac{3}{2}} b, \quad \nu = \frac{n}{E},$$

we have:

$z = \alpha$ is a root of $H_{m,n}(E^{\frac{1}{2}} z)$ if and only if $\exists \beta$ such that

$$\psi''(\lambda) = \left(E^2 V(\lambda; \alpha, \beta, \nu) - \frac{1}{4\lambda^2} \right) \psi(\lambda),$$

$$V(\lambda; \alpha, \beta, \nu) = \lambda^2 + 2\alpha\lambda + \alpha^2 - 1 - \beta\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2},$$

satisfies **apparent singularity** and **quantisation condition**.

Next step: complex WKB approach as $E \to \infty$.

Complex WKB Approach

As $E \to \infty$ solutions of anharmonic oscillator are well-approximated by WKB functions

$$\psi = V^{-\frac{1}{4}} e^{\pm E \int^{\lambda} \sqrt{V(\mu)} d\mu},$$

$$V = \lambda^2 + 2\boldsymbol{\alpha}\lambda + \boldsymbol{\alpha}^2 - 1 - \boldsymbol{\beta}\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2}.$$

This yields, as $E \to \infty$, that the **apparent singularity** and **quantisation condition** are asymptotically equivalent to a set of conditions,

- one geometric,
- two analytic,

on the potential $V = V(\lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}, \nu)$.

Stokes Geometry

Consider potential

$$V(\lambda;\boldsymbol{\alpha},\boldsymbol{\beta},\nu) = \lambda^2 + 2\boldsymbol{\alpha}\lambda + \boldsymbol{\alpha}^2 - 1 - \boldsymbol{\beta}\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2}.$$

- **Stokes lines** are level sets $\Re \int_{\lambda^*}^{\lambda} \sqrt{V(\lambda)}d\lambda = 0$ in $\mathbb{P}^1$, where λ^* any zero of $V(\lambda)$.
- **Stokes complex** $\mathcal{C} = \mathcal{C}(\boldsymbol{\alpha},\boldsymbol{\beta}) \subseteq \mathbb{P}^1$ of $V(\lambda)$ is union of all its Stokes lines and zeros.

Geometric Condition on Potential

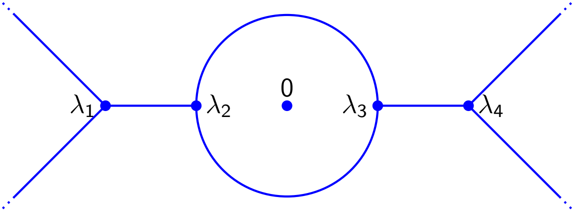


Figure: Stokes complex $\mathcal{C}(\alpha, \beta)$ with $(\alpha, \beta) = (0, 0)$, where $\lambda_{1,2,3,4}$ are the zeros of $V = \lambda^2 - 1 + \frac{\nu^2}{4}\lambda^{-2}$.

Geometric Condition on potential $V(\lambda; \alpha, \beta, \nu)$

The Stokes complex $\mathcal{C}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of $V(\lambda)$ is homeomorphic to the Stokes complex $\mathcal{C}(0, 0)$.

A Pair of Cycles

Consider elliptic curve

$$p^2 = \lambda^2 V(\lambda) = \lambda^4 + 2\alpha\lambda^3 + (\alpha^2 - 1)\lambda^2 - \beta\lambda + \frac{\nu^2}{4}.$$

Assume $V(\lambda; \alpha, \beta, \nu)$ satisfies the **geometric condition**, then we can rigidly define two cycles $\gamma_{1,2}$ on elliptic curve as in figure.

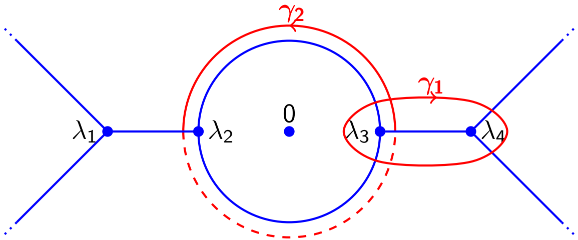


Figure: Cycles $\gamma_{1,2}$ on elliptic curve where γ_1 lies in sheet $p \sim +\frac{\nu}{2}$ as $\lambda \to 0$.

Two Complete Elliptic Integrals

Let $\omega := \frac{p}{\lambda} d\lambda$ be pull-back of $\sqrt{V} d\lambda$ on elliptic curve

$$p^2 = \lambda^4 + 2\boldsymbol{\alpha}\lambda^3 + (\boldsymbol{\alpha}^2 - 1)\lambda^2 - \boldsymbol{\beta}\lambda + \frac{\nu^2}{4}.$$

Assume $V(\lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}, \nu)$ satisfies the **geometric condition**, then we define the complete elliptic integrals

$$s_1(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_{\boldsymbol{\gamma_1}} \omega + \frac{i\pi(1-\nu)}{2},$$
$$s_2(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_{\boldsymbol{\gamma_2}} \omega.$$

WKB Result (heuristically)

As E grows large,

- The **quantisation condition** is asymptotically equivalent to quantisation

$$s_1(\alpha, \beta) = i\frac{\pi j}{E}, \quad j \in I_m := \{-m+1, -m+3, \ldots, +m-1\}. \tag{2}$$

- The **apparent singularity condition** is asymptotically equivalent to quantisation

$$s_2(\alpha, \beta) = i\frac{\pi k}{E}, \quad k \in I_n := \{-n+1, -n+3, \ldots, +n-1\}. \tag{3}$$

Note: (2) and (3) are classical Bohr-Sommerfeld quantisation conditions.

Accounting: $\#(I_m \times I_n) = m \times n = \deg H_{m,n}$.

Results

Elliptic Region

Definition (Elliptic Region)

Let R be region where geometric condition on Stokes complex is satisfied,

$$R = \{(\alpha, \beta) \in \mathbb{C}^2 : \mathcal{C}(\alpha, \beta) \cong \mathcal{C}(0,0)\}.$$

Denote its closure by $K = \overline{R}$ and let $K_a = K_a(\nu)$ be the projection of K onto α-plane. We call K_a the **elliptic region**.

Theorem (Elliptic Region, part 1)

As $E \to \infty$, roots of $H_{m,n}(E^{\frac{1}{2}} z)$ **densely fill** *up elliptic region K_a.*

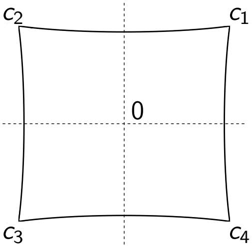
Elliptic Region, Corners

Theorem (Elliptic Region, part 2)

The elliptic region K_a is a compact quadrilateral domain whose boundary is a Jordan curve composed of four analytic pieces (edges), meeting at four corners $c_{1,2,3,4}$, as in figure.
The corner c_k is the unique solution of

$$\alpha^8 - 6(3\nu^2 + 1)\alpha^4 + 8(1 - 9\nu^2)\alpha^2 - 3(9\nu^4 + 6\nu^2 + 1) = 0$$

in k-th quadrant of complex α-plane. (Remaining four roots are purely real or imaginary)



Elliptic Region, Boundary Parametrisation

Theorem (Elliptic Region, part 3)

Cut α-plane along diagonals $[c_1, c_3]$ and $[c_2, c_4]$. Then

$$\psi(\alpha) = \tfrac{1}{2}\Re\left[\alpha y + \tfrac{1}{2}(1-\nu)\log(p_1) - \log(p_2) + \nu\log(x^{-2}p_3)\right],$$

$$p_1 = 1 - 2x\alpha - 2x^2, \quad p_2 = 2x + \alpha + y, \quad p_3 = x(\alpha^2 + 5x\alpha + 4x^2 - 1) + \frac{1}{2}\nu y,$$

is a **univalued harmonic function** *on this cut plane.*

Here $x = x(\alpha)$ and $y = y(\alpha)$ are the unique algebraic functions which solve

$$3x^4 + 4\alpha x^3 + (\alpha^2 - 1)x^2 - \frac{\nu^2}{4} = 0, \qquad x(\alpha) \sim \frac{\nu}{2}\alpha^{-1} \quad (\alpha \to \infty),$$

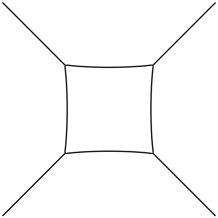
$$y^2 = \alpha^2 + 6x\alpha + 6x^3 - 1, \qquad y(\alpha) \sim \alpha \quad (\alpha \to \infty),$$

on the same cut plane.

Elliptic Region, Boundary Parametrisation

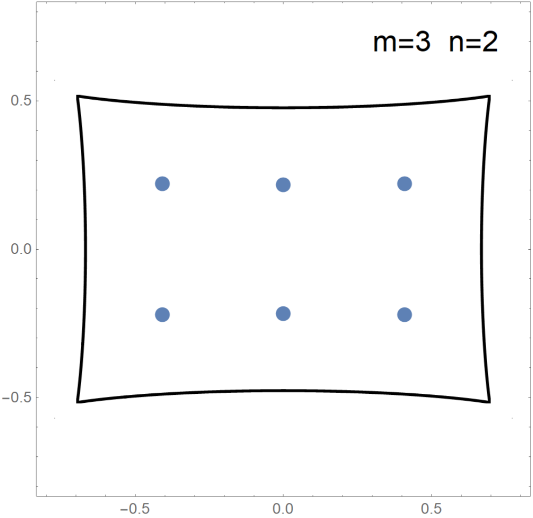
Theorem (Elliptic Region, part 4)

The level set $\{\psi(\alpha) = 0\}$ *consists of* **boundary elliptic region** ∂K_a *plus four additional lines which emanate from corners and go to infinity, see figure.*



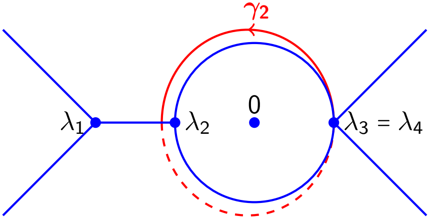
Buckingham (2018) obtained different parametrisation elliptic region via Riemann-Hilbert approach to certain orthogonal polynomials and proved: asymptotically there are **no roots outside elliptic region**.

The Elliptic Region, $\nu = \frac{1}{4}, \ \frac{m}{n} = \frac{3}{2}$

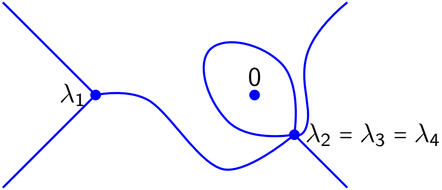


Geometric meaning boundary elliptic region

Geometrically right-edge of elliptic region K_a is characterised by coalescence two zeros $\lambda_{3,4}$ of potential:



Similarly, top-right corner is characterised by coalescence of three zeros $\lambda_{2,3,4}$ of potential:



Asymptotic distribution within K_a

Recall Bohr-Sommerfeld quantisation conditions

$$s_1(\alpha, \beta) = i\frac{\pi j}{E}, \quad j \in \{-m+1, -m+3, \ldots, +m-1\}, \tag{4}$$

$$s_2(\alpha, \beta) = i\frac{\pi k}{E}, \quad k \in \{-n+1, -n+3, \ldots, +n-1\}. \tag{5}$$

We may eliminate β by imposing $\Re s_{1,2}(\alpha, \beta) = 0$.
Then equations

$$\Im s_1(\alpha) = \frac{\pi j}{E}, \quad j \in \{-m+1, -m+3, \ldots, +m-1\}$$

define m 'vertical' grid lines within K_a.
Similarly equations

$$\Im s_2(\alpha) = \frac{\pi k}{E}, \quad k \in \{-n+1, -n+3, \ldots, +n-1\}$$

define n 'horizontal' grid lines within K_a.

Deformed rectangular lattice

Example $(m,n) = (4,3)$:

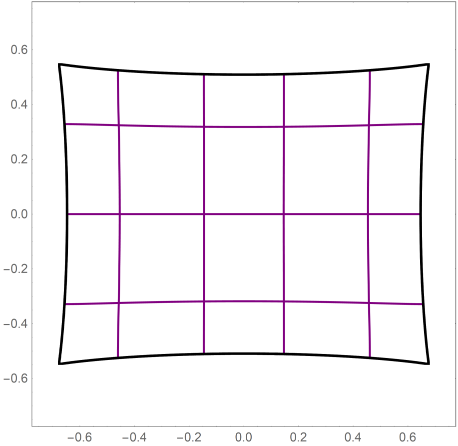


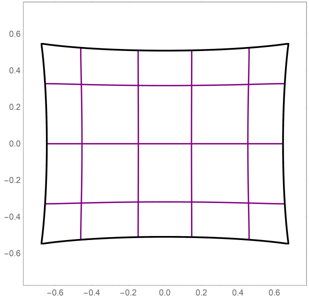
Figure: **deformed rectangular lattice** within elliptic region K_a

Asymptotic distribution within K_a

Theorem (Asymptotic Distribution of Bulk, heuristically)

In the large E limit, the bulk of the (rescaled) roots organise themselves within elliptic region K_a along the **vertices** *of* **deformed rectangular lattice** *defined by Bohr-Sommerfeld quantisation conditions.*

Example $(m, n) = (4, 3)$:



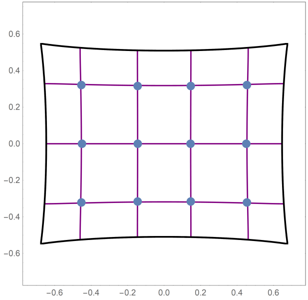


Figure: In both figures **deformed rectangular lattice** with on the right true locations of **roots superimposed**.

Asymptotic Distribution

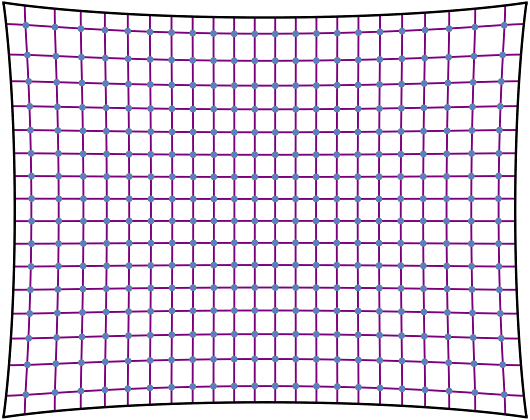
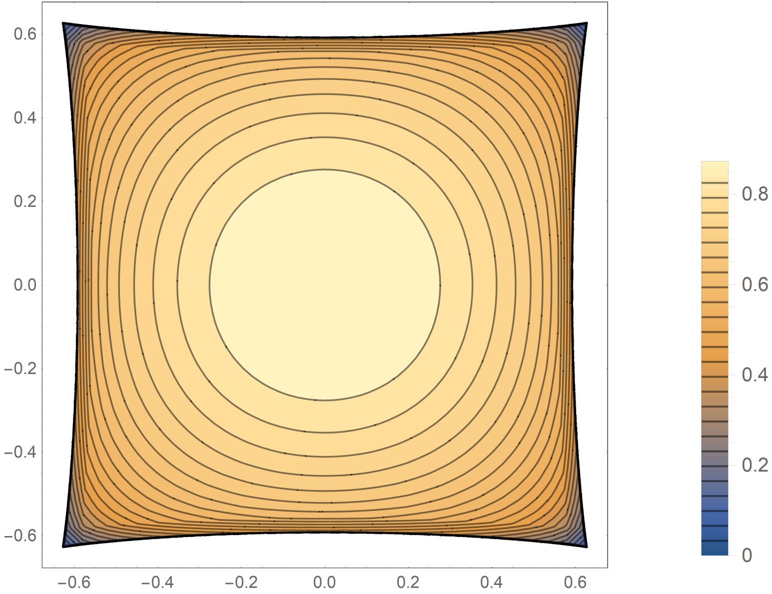


Figure: Asymptotic prediction are vertices of purple lattice, true location roots $H_{m,n}(z)$ in blue, with $(m, n) = (22, 16)$.

Asymptotic root-density plot, $\nu = \frac{1}{3}$

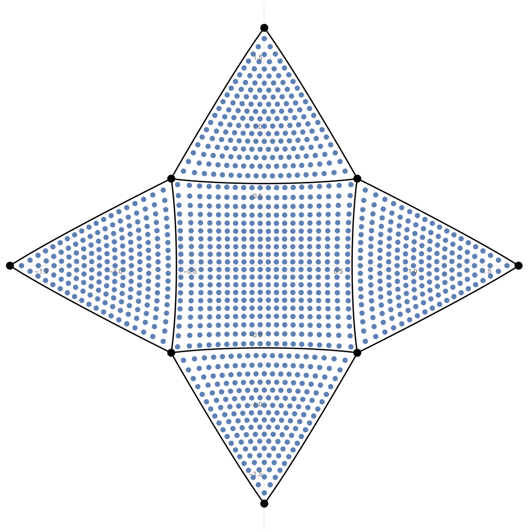


Future

Painlevé equation	anharmonic oscillator	rational solutions
P_{VI}	Heun	**Jacobi**
P_{V}	confluent Huen	**Laguerre**
P_{IV}	biconfluent Heun	**Hermite**, **Okamoto**
P_{III}	doubly confluent Heun	**Umemura**
P_{II}	triconfluent Heun	**Yablonskii-Vorob'ev**
P_{I}	cubic oscillator	**none**

Blue: open
Red: done

Preliminary Result



$E \to \infty,\ n = \mathcal{O}(1).$

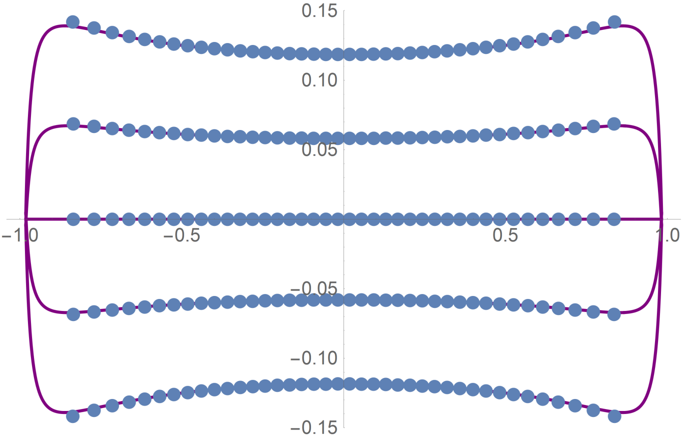


Figure: $(m, n) = (40, 5)$

Explicit formula $s_{1,2}$

$$s_1 = +\frac{2i}{\sqrt{(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)}}F(\lambda_1,\lambda_2,\lambda_3,\lambda_4) + \frac{1}{2}i\pi(1-\nu),$$
$$s_2 = -\frac{2}{\sqrt{(\lambda_4-\lambda_3)(\lambda_2-\lambda_1)}}F(\lambda_4,\lambda_1,\lambda_2,\lambda_3) + i\pi\nu,$$

with

$$\begin{aligned} F(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = &-\tfrac{1}{4}(\lambda_4-\lambda_2)(\lambda_3-\lambda_2)(3\lambda_1-\lambda_2+\lambda_3+\lambda_4)\mathcal{K}(m) \\ &+\tfrac{1}{4}(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\mathcal{E}(m) \\ &+(\lambda_4-\lambda_2)\Pi(n_1,m)+2\lambda_1\lambda_3(\lambda_4-\lambda_2)\Pi(n_2,m), \end{aligned}$$

where

$$m = \frac{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_2)}, \quad n_1 = -\frac{\lambda_4-\lambda_3}{\lambda_3-\lambda_2}, \quad n_2 = -\frac{(\lambda_4-\lambda_3)\lambda_2}{(\lambda_3-\lambda_2)\lambda_4},$$

and

$$\lambda^4 + 2\boldsymbol{\alpha}\lambda^3 + (\boldsymbol{\alpha}^2-1)\lambda^2 - \boldsymbol{\beta}\lambda + \frac{\nu^2}{4} = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4).$$