On some inverse problems related to Painlevé functions

Pieter Roffelsen

The University of Sydney

September, 2022

Applicable Resurgent Asymptotics II
Isaac Newton Institute

Plan seminar

Theory:

- Introduction to Painlevé equations
- Monodromy manifolds
- Recent results on $q\mathrm{P}_{\mathrm{VI}}$
- (Implicit) asymptotics

Based on joint work with Nalini Joshi (University of Sydney)

Practice:

- Rational solutions of Painlevé IV
- Poles of solutions and biconfluent Heun equations
- Asymptotic pole distributions via WKB

Based on joint work with Davide Masoero (University of Lisbon)

Classical Painlevé equations

The classical Painlevé equations, P_{I},...,P_{VI}, are 2nd order **nonlinear** ODEs in the complex plane,

$$u_{tt} = R(u, u_t, t),$$

with R rational, without **movable** branch points.
That is, for any (local) parametrisation of the solution space,

$$u(t) = u(t;\eta),$$

the locations of the branch points of u are independent of η.

P_{I}, the simplest to write down, $u_{tt} = 6u^2 - t$.
P_{VI}, the most involved:

$$\begin{aligned} u_{tt} = &\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\frac{u_t^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u_t \\ &+ \frac{u(u-1)(u-t)}{2t^2(t-1)^2}\left((\theta_\infty - 1)^2 - \frac{\theta_0^2 t}{u^2} + \frac{\theta_1^2(t-1)}{(u-1)^2} + \frac{t(t-1)(1-\theta_t^2)}{(u-t)^2}\right), \end{aligned}$$

where $\theta = (\theta_0, \theta_t, \theta_1, \theta_\infty)$ complex parameters.

Parametrising solution spaces

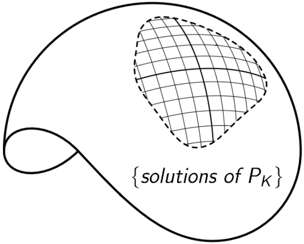
How to parametrise the solution space of a Painlevé equation?

- A local method: fix a point t_0 in the complex plane and specify

$$u(t_0) = \eta_1, \quad u'(t_0) = \eta_2, \quad (\eta_1, \eta_2) \in \mathbb{C}^2.$$

  This does not cover full solution space, as e.g. $u(t)$ can have a pole at $t = t_0$. Okamoto (1979) constructed full spaces of initial conditions for the Painlevé equations.

- A global method: via the **Riemann-Hilbert correspondence**.



Monodromy manifolds

- Each Painlevé equation P_K, $K = \mathrm{I}, \dots \mathrm{VI}$, is integrable: it has an associated linear system

$$Y_z = A_K(z; u, u_t, t)Y,$$

such that, as t moves, the **monodromy data** of the linear system are preserved. [Flaschka and Newell 1980, Jimbo et al 1981]

- The **monodromy data** form a complete set of **first integrals**.
- This yields a one-to-one correspondence

$$\textit{solutions of } P_K \leftrightarrow \textit{monodromy data}.$$

- The collection of monodromy data

$$M_K = \{\textit{monodromy data}\},$$

is called the corresponding **monodromy manifold**.

Monodromy manifolds as algebraic surfaces

Each of these monodromy manifolds M_K can be identified with an **affine cubic surface**

$$M_K \cong \{\eta \in \mathbb{C}^3 : R_K(\eta) = 0\} \qquad (R_K \text{ a cubic polynomial}).$$

Therefore, we have a (generically) one-to-one correspondence

$$\{\textit{solutions of } P_K\} \to \{\eta \in \mathbb{C}^3 : R_K(\eta) = 0\}.$$

P-eqs	polynomials
P_{VI}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + \eta_3^2 + w_1\eta + w_2\eta_2 + w_3\eta_3 + w_4$
P_{V}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + R(w_{1,2,3})$
$P_{\mathrm{V}}^{\mathrm{deg}}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_1 - 1$
P_{IV}	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 + w_2(\eta_2 + \eta_3) + w_2(1 + w_1 - w_2)$
$P_{\mathrm{III}}^{D_6}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_1 - 1$
$P_{\mathrm{III}}^{D_7}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 - \eta_2$
$P_{\mathrm{III}}^{D_8}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 - \eta_2$
$P_{\mathrm{II}}^{\mathrm{JM}}$	$\eta_1\eta_2\eta_3 + \eta_1^2 + w_1\eta_1 - \eta_2 - 1$
$P_{\mathrm{II}}^{\mathrm{FN}}$	$\eta_1\eta_2\eta_3 - \eta_1 + w_2\eta_2 - \eta_3 - w_2 + 1$
P_{I}	$\eta_1\eta_2\eta_3 - \eta_1 - \eta_2 + 1$

See e.g. van der Put and Saito (2009) and Chekhov et al. (2015).

Example: Painlevé I

- Nonlinear equation: $P_{\mathrm{I}} : u_{tt} = 6u^2 - t$.
- The corresponding (scalar) linear problem:

$$Y_{zz} = V(z; u, u_t, t)\, Y,$$

$$V = 4z^3 - 2t\,z + 2t\,u - 4u^3 + u_t^2 + \frac{u_t}{z-u} + \frac{3}{4(z-u)^2}.$$

- This is **the cubic oscillator**+apparent singularity at $z = u$.
- Monodromy data are given by Stokes multipliers near irregular singularity at $z = \infty$.

Stokes Phenomenon

Canonical solutions are defined by subdominant asymptotic behaviour

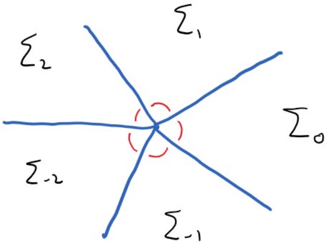
$$y_k(z) = \begin{cases} \frac{1}{2} z^{-\frac{3}{4}} \exp\left[-\frac{4}{5} z^{\frac{5}{2}} + t\, z^{\frac{1}{2}}\right]\left(1 + \mathcal{O}(z^{-\frac{1}{2}})\right) & \text{if } k \text{ is even,} \\ \frac{1}{2} z^{-\frac{3}{4}} \exp\left[+\frac{4}{5} z^{\frac{5}{2}} - t\, z^{\frac{1}{2}}\right]\left(1 + \mathcal{O}(z^{-\frac{1}{2}})\right) & \text{if } k \text{ is odd,} \end{cases}$$

as $z \to \infty$, in Stokes sector Σ_k, $k \in \mathbb{Z}$.
Stokes phenomenon is then given by

$$y_{k+1}(z) = y_{k-1}(z) + s_k y_k(z),$$

where Stokes multipliers s_k satisfy

$$s_{k+5} = s_k, \quad 1 + s_k s_{k+1} = -i s_{k+3} \quad (k \in \mathbb{Z}).$$



The P_{I} cubic surface

The Stokes multipliers (s_{-1}, s_0, s_1) satisfy the **cubic** equation

$$s_1 s_0 s_{-1} + s_1 + s_{-1} + i = 0.$$

Solutions of P_{I} are in one-to-one correspondence with triples $(s_{-1}, s_0, s_1) \in \mathbb{C}^3$ satisfying this cubic.

Why should you care?

Geometry of cubic surface is intricately related to asymptotics of solutions!

For example, the affine cubic surface has precisely 5 **lines**:

$$\begin{array}{llll}
\mathcal{L}_2: & s_1 = \lambda, & s_0 = -i, & s_{-1} = -i, \\
\mathcal{L}_1: & s_1 = 0, & s_0 = \lambda, & s_{-1} = -i, \\
\mathcal{L}_0: & s_1 = \lambda, & s_0 = 0, & s_{-1} = -i - \lambda, \\
\mathcal{L}_{-1}: & s_1 = -i, & s_0 = \lambda, & s_{-1} = 0, \\
\mathcal{L}_{-2}: & s_1 = -i, & s_0 = -i, & s_{-1} = \lambda.
\end{array}$$

Each line corresponds to a one-parameter family of **tronquée solutions**!

Intersection points and Tritronquée

On $\mathcal{L}_2$, the solutions $u(t)$ satisfy

$$u(t) \sim -\sqrt{t/6} \quad (t \to \infty, 0 < \arg t < \frac{4\pi}{5}).$$

On $\mathcal{L}_{-2}$, the solutions $u(t)$ satisfy

$$u(t) \sim -\sqrt{t/6} \quad (t \to \infty, -\frac{4\pi}{5} < \arg t < 0).$$

The lines $\mathcal{L}_2$ and $\mathcal{L}_{-2}$ intersect at a single point,

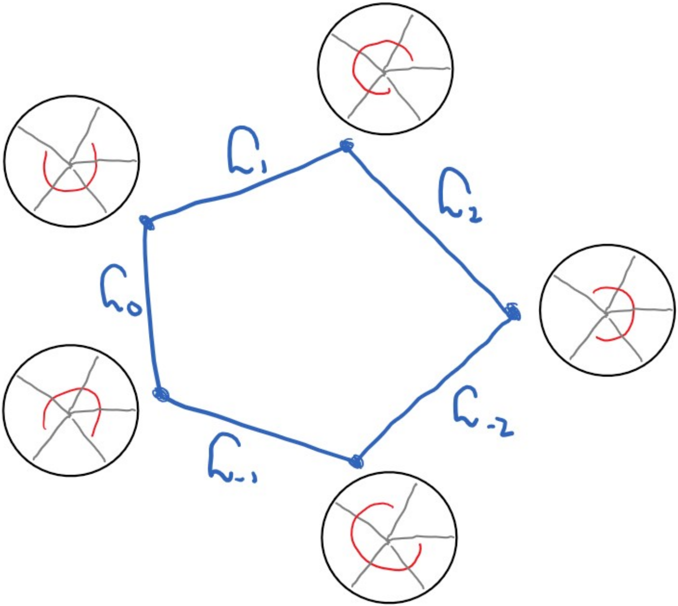
$$(s_{-1}, s_0, s_1) = (-i, -i, -i).$$

The corresponding solution is the **tritronquée** solution $u_*(t)$, uniquely defined by

$$u_*(t) \sim -\sqrt{t/6} \quad (t \to \infty, -\frac{4\pi}{5} < \arg t < \frac{4\pi}{5}).$$

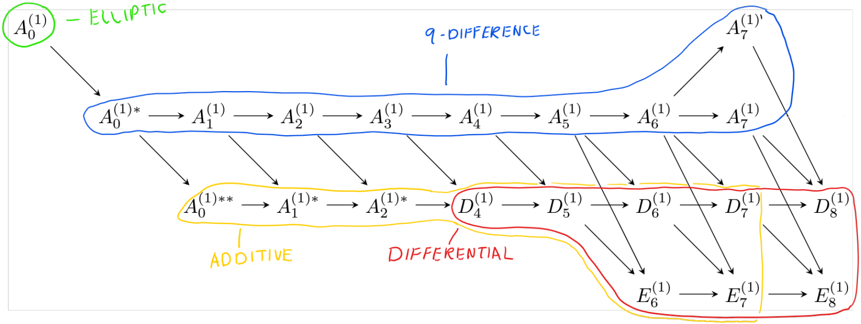
References: Boutroux (1913), Joshi and Kruskal (1988), Kapaev and Kitaev (1993), Takei (1995), Joshi and Kitaev (2001).

Configuration of lines



Discrete Painlevé equations

Sakai (2001) classified discrete Painlevé equations according to surface type.



- Within green: elliptic Painlevé.
- Within blue: q-difference Painlevé.
- Within yellow: additive Painlevé.
- Within red: differential Painlevé.

Riemann-Hilbert correspondence for discrete Painlevé?

Open problem

What algebraic surfaces are on the right-hand side of the Riemann-Hilbert correspondence for discrete Painlevé equations?

P-eqs	polynomials
$\vdots$	$\vdots$
$eP(A_0^{(1)})$	?
$qP(A_3^{(1)})$	?
$qP(A_6^{(1)})$	?
$dP(A_2^{(1)*})$	?
P_{VI}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + \eta_3^2 + w_1\eta + w_2\eta_2 + w_3\eta_3 + w_4$
P_{V}	$\eta_1\eta_2\eta_3 + \eta_1^2 + \eta_2^2 + w_1\eta_1 + w_2\eta_2 + w_3\eta_3 + R(w_{1,2,3})$
$\vdots$	$\vdots$
$P_{\mathrm{II}}^{\mathrm{FN}}$	$\eta_1\eta_2\eta_3 - \eta_1 + w_2\eta_2 - \eta_3 - w_2 + 1$
P_{I}	$\eta_1\eta_2\eta_3 - \eta_1 - \eta_2 + 1$

Literature: Chekhov, Mazzocco, Rubtsov (2020) conjectured certain affine cubic surfaces for $e\mathrm{P}(A_0^{(1)})$ and $q\mathrm{P}(A_k^{(1)})$, $k = 0, 1, 2$.

q-Painlevé VI

Fix $q \in \mathbb{C}$ with $0 < |q| < 1$. Then q-Painlevé VI is given by

$$q\mathrm{P}_{\mathrm{VI}}: \begin{cases} f\overline{f} &= \dfrac{(\overline{g} - \kappa_0 t)(\overline{g} - \kappa_0^{-1} t)}{(\overline{g} - \kappa_\infty)(\overline{g} - q^{-1}\kappa_\infty^{-1})}, \\ g\overline{g} &= \dfrac{(f - \kappa_t t)(f - \kappa_t^{-1} t)}{q(f - \kappa_1)(f - \kappa_1^{-1})}, \end{cases}$$

where

- $f, g : T \to \mathbb{CP}^1$, with $T = q^{\mathbb{Z}} t_0$.
- $f = f(t)$, $g = g(t)$, $\overline{f} = f(qt)$, $\overline{g} = g(qt)$, for $t \in T$.
- $\kappa = (\kappa_0, \kappa_t, \kappa_1, \kappa_\infty) \in \mathbb{C}^4$ are nonzero complex parameters.

Theorem, PR and Joshi (2022)

The answer for $q\mathrm{P}_{\mathrm{VI}}(A_3^{(1)})$ is:

an affine **Segre surface**.

Segre surfaces

P-eq	polynomials
q-P_{VI}	$R_1 = u_0\eta_1^2 + u_1\eta_1\eta_2 + u_2\eta_1\eta_3 + u_3\eta_1\eta_4 + u_4\eta_3\eta_4 + u_5\eta_1$ $R_2 = v_0\eta_2^2 + v_1\eta_1\eta_2 + v_2\eta_2\eta_3 + v_3\eta_2\eta_4 + v_4\eta_3\eta_4 + v_5\eta_2$

A Segre surface is by definition the intersection of two quadrics in $\mathbb{CP}^4$,

$$\{\eta \in \mathbb{CP}^4 : P(\eta) = 0, Q(\eta) = 0\},$$

for quadratic polynomials P, Q.
They were introduced by Corrado Segre (1884), who showed that they always have 16 lines (counting multiplicity), amongst other things.

Explicit formulas for coefficients

$$u_0 = \kappa_\infty^2 \theta_q\left(\kappa_t^2, \kappa_1^2, t_0\kappa_t\kappa_1, \frac{t_0}{\kappa_t\kappa_1\kappa_\infty^2}\right), \qquad u_1 = \theta_q\left(\kappa_t^2\kappa_1^2, \kappa_\infty^2, \frac{t_0\kappa_1}{\kappa_t}, \frac{t_0\kappa_t}{\kappa_1}\right),$$

$$u_2 = \kappa_t^2 \theta_q\left(\frac{\kappa_1\kappa_\infty}{\kappa_0\kappa_t}, \frac{\kappa_0\kappa_1\kappa_\infty}{\kappa_t}, \frac{t_0}{\kappa_t\kappa_1}, t_0\kappa_t\kappa_1\right), \quad u_3 = -\theta_q\left(\kappa_t^2\kappa_\infty^2, \kappa_1^2, \frac{t_0}{\kappa_t\kappa_1}, t_0\kappa_t\kappa_1\right),$$

$$u_4 = \theta_q\left(\kappa_t^2, \kappa_1^2, \frac{t_0}{\kappa_t\kappa_1}, t_0\kappa_t\kappa_1\kappa_\infty^2\right), \qquad u_5 = \frac{\theta_q(\kappa_0\kappa_t\kappa_1\kappa_\infty)}{\theta_q(\kappa_0)^2}\theta_q\left(\frac{\kappa_t\kappa_1\kappa_\infty}{\kappa_0}\right),$$

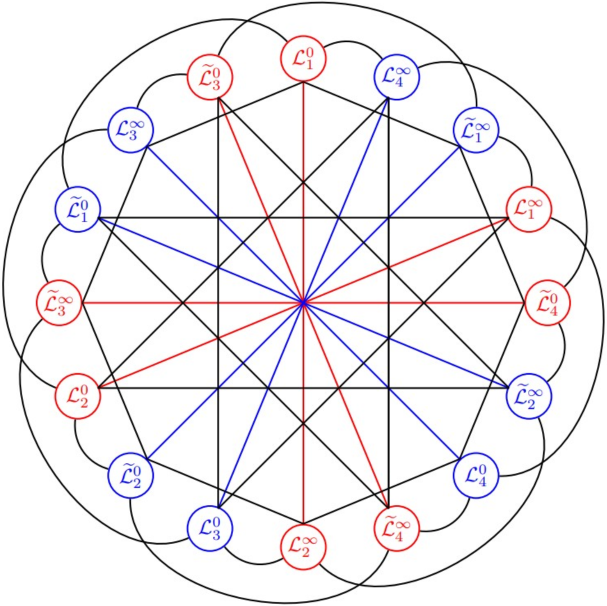
and

$$v_0 = \theta_q\left(t_0\kappa_t\kappa_1, \frac{t_0\kappa_\infty^2}{\kappa_t\kappa_1}, \frac{t_0\kappa_t}{\kappa_1}, \frac{t_0\kappa_1}{\kappa_t}\right), \qquad v_1 = -\frac{t_0}{\kappa_t\kappa_1}\theta_q\left(t_0^2, \kappa_t^2, \kappa_1^2, \kappa_\infty^2\right),$$

$$v_2 = -\theta_q\left(\frac{t_0\kappa_t}{\kappa_1}, \frac{t_0\kappa_1\kappa_\infty^2}{\kappa_t}, t_0\kappa_t\kappa_1, \frac{t_0}{\kappa_t\kappa_1}\right), \quad v_3 = -\theta_q\left(\frac{t_0\kappa_1}{\kappa_t}, \frac{t_0\kappa_t\kappa_\infty^2}{\kappa_1}, t_0\kappa_t\kappa_1, \frac{t_0}{\kappa_t\kappa_1}\right)$$

$$v_4 = \theta_q\left(\frac{t_0}{\kappa_t\kappa_1}, t_0\kappa_t\kappa_1\kappa_\infty^2, \frac{t_0\kappa_t}{\kappa_1}, \frac{t_0\kappa_1}{\kappa_t}\right), \qquad v_5 = \frac{t_0}{\kappa_t\kappa_1}\frac{\theta_q(t_0\kappa_0\kappa_\infty)}{\theta_q(\kappa_0)^2}\theta_q\left(\frac{t_0\kappa_\infty}{\kappa_0}\right).$$

Configuration of lines



Part 2: Practice

We can fix a Painlevé function by specifying the equation, the parameters and monodromy. For example, in the case of Painlevé I, the general solution is given by

$$u_{\mathrm{I}}(t) = u_{\mathrm{I}}(t; s_{-1}, s_0, s_1),$$

with the triple (s_{-1}, s_0, s_1) of coordinates satisfying

$$s_1 s_0 s_{-1} + s_1 + s_{-1} + i = 0.$$

How to compute u_{I}?

Standard inverse problem

Fix Stokes data (s_{-1}, s_0, s_1), take a point $t \in \mathbb{C}$, compute $u_{\mathrm{I}}(t)$.

This is equivalent to finding $(u_*, v_*) \in \mathbb{C}^2$ such that the Stokes data of

$$Y_{zz} = V(z; u_*, v_*, t) Y,$$

$$V = 4z^3 - 2t\,z + 2t\,u_* - 4u_*^3 + v_*^2 + \frac{v_*}{z - u_*} + \frac{3}{4(z - u_*)^2},$$

are given by (s_{-1}, s_0, s_1). Then $u_I(t) = u_*$ and $u_I'(t) = v_*$.

Alternative in the spirit of Nevanlinna

Alternative inverse problem

Fix Stokes data (s_{-1}, s_0, s_1) and a value $w \in \mathbb{CP}^1$. Compute the set

$$u_{\mathrm{I}}^{-1}(w) = \{t \in \mathbb{C} : u_{\mathrm{I}}(t) = w\}.$$

Important observation: this problem simplifies for $w = \infty$!

Theorem (Masoero (2009))

Fix Stokes data (s_{-1}, s_0, s_1) and a corresponding solution $u_{\mathrm{I}}(t)$. Then $t = a$ is a pole of $u_I(t)$ if and only if $\exists b \in \mathbb{C}$ such that the **cubic oscillator**

$$Y_{zz} = V(z; a, b) Y,$$
$$V = 4z^3 - 2a\, z + b,$$

has Stokes data given by (s_{-1}, s_0, s_1).

Poles of rational P_{IV} functions

For $m, n \in \mathbb{N}$,

$$u_{m,n} = \frac{H'_{m+1,n}}{H_{m+1,n}} - \frac{H'_{m,n}}{H_{m,n}}, \quad \theta_0 = \tfrac{1}{2}n, \quad \theta_\infty = m + \tfrac{1}{2}n + 1,$$

defines a rational solution of the fourth Painlevé equation

$$\mathrm{P}_{\mathrm{IV}}: \quad u_{tt} = \frac{1}{2u}u_t^2 + \tfrac{3}{2}u^3 + 4tu^2 + 2(t^2 + 1 - 2\theta_\infty)u - \frac{8\theta_0^2}{u}.$$

Here the $H_{m,n}$ are generalised Hermite polynomials,

$$H_{m,n} = \mathcal{W}(h_m, h_{m+1}, \dots, h_{m+n-1}) \quad (m, n \in \mathbb{N}),$$

where h_k denotes k-th Hermite polynomial. (Noumi and Yamada 1999)

Note: poles of rational solutions coincide with roots of generalised Hermite polynomials!

Generalised Hermite Polynomials

Degree:

$$\deg(H_{m,n}) = m \times n$$

Symmetry:

$$H_{n,m}(z) = i^{mn} H_{m,n}(-iz)$$

Examples:

$$H_{m,1}(z) = h_m(z) \quad (m \in \mathbb{N}),$$
$$H_{2,2}(z) = z^4 + 12$$
$$H_{3,2}(z) = z^6 - 6z^4 + 36z^2 + 72$$
$$H_{3,3}(z) = z^9 + 72z^5 - 2160z$$
$$H_{4,2}(z) = z^8 - 16z^6 + 120z^4 + 720$$
$$H_{4,3}(z) = z^{12} - 12z^{10} + 180z^8 - 480z^6 - 3600z^4 - 43200z^2 + 43200$$
$$H_{4,4}(z) = z^{16} + 240z^{12} - 7200z^8 + 2016000z^4 + 6048000$$

Roots of generalised Hermite polynomials

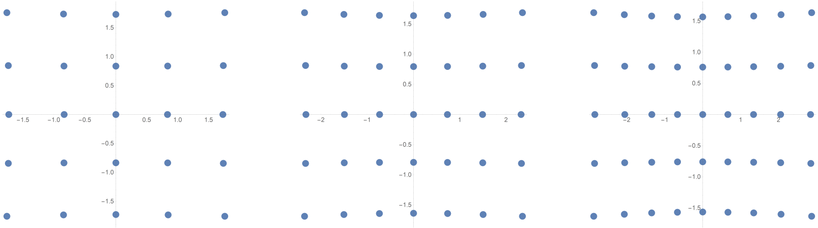


Figure: Roots of $H_{m,n}$, with $n = 5$ and $m = 5, 7, 9$

Problem (Clarkson, 2003)

The roots seem to lie on a deformed rectangular lattice. Is there an analytic description of the roots explaining this regularity?

We will consider this problem asymptotically as $m, n \to \infty$, with $m/n = \mathcal{O}(1)$.

Method of attack

- Note that problem of computing root distributions is precisely in the alternative inverse problem form (à la Nevanlinna).
- Apply integrability of P_{IV} to obtain characterisation in terms of biconfluent Heun equations.
- Characterise these Heun equations in terms of classes of Nevanlinna-Belyi functions.
- Use complex WKB approach to obtain asymptotic descriptions of distributions of roots

Classification

Theorem (D. Masoero and PR, 2018)

For $m, n \in \mathbb{N}$, the point $a \in \mathbb{C}$ is a root of $H_{m,n}$ if and only if there exists an (a fortiori unique) $b \in \mathbb{C}$ such that the biconfluent Heun equation

$$\psi''(z) = \left(z^2 + 2a\,z + a^2 - (2m+n) - \frac{b}{z} + \frac{n^2-1}{4z^2}\right)\psi(z), \qquad (1)$$

satisfies the following conditions:

1. **Apparent Singularity Condition.** *The monodromy around Fuchsian singularity $z = 0$ is scalar. In a formula,*

$$\psi(e^{2\pi i} z) = (-1)^{n+1}\psi(z)\,, \quad \forall \psi \text{ solution of } (1).$$

2. **Quantisation Condition.** *There exists a non-zero solution of (1) which solves the following boundary value problem*

$$\lim_{z \to +\infty} \psi(z) = \lim_{z \to 0^+} \psi(z) = 0\,.$$

Nevanlinna-Belyi function characterisation

Take $m, n \in \mathbb{N}$ and let $(a, b) \in \mathbb{C}^2$ be such that

$$\psi''(z) = (z^2 + 2a\,z + a^2 - (2m+n) - \frac{b}{z} + \frac{n^2-1}{4z^2})\psi(z),$$

satisfies the apparent singularity and quantisation condition.
Then any ratio of solutions $f = \psi/\phi : \mathbb{C} \to \mathbb{CP}^1$ defines a meromorphic function, which satisfies

- f has a critical point of order $n-1$ at $z = 0$,
- In each of the four Stokes sectors,

$$\Sigma_k = \{z \in \mathbb{Z} : |\arg z - \frac{k\pi}{2}| < \frac{\pi}{2}\}, \quad k \in \mathbb{Z}_4,$$

  f has a critical points of infinite order, technically a *logarithmic direct transcendental singularity*. Namely,

$$f(z) \to w_k, \quad (z \to \infty, z \in \Sigma_k),$$

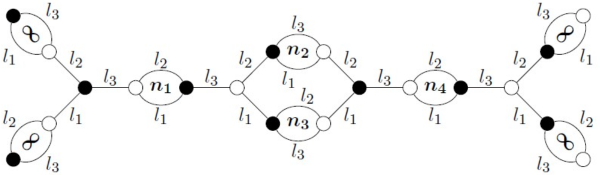
  exponentially fast, for some $w_k \in \mathbb{CP}^1$.

Nevanlinna-Belyi function characterisation, continued

- The critical value $f(0) = w_*$, at $z = 0$, and the critical values at $z = \pm\infty$ coincide,

$$w_* = w_0 = w_2.$$

So f is a branched covering ramified over only three points (Belyi function).
It sits in a larger class of meromorphic functions introduced and studied by Nevanlinna (1932), and his student Elfving (1934).
its Dessin d'enfant is given by



for some $(n_1, n_2, n_3, n_4) \in \mathbb{N}^4$ with $n_1 + n_4 = m - 1$, $n_2 + n_3 = n - 1$.

Rescaling

Setting

$$E = 2m + n, \quad \boldsymbol{\alpha} = E^{-\frac{1}{2}} \boldsymbol{a}, \quad \boldsymbol{\beta} = E^{-\frac{3}{2}} \boldsymbol{b}, \quad \nu = \frac{n}{E},$$

we have:

$t = \boldsymbol{\alpha}$ is a root of $H_{m,n}(E^{\frac{1}{2}} t)$ if and only if $\exists \boldsymbol{\beta}$ such that

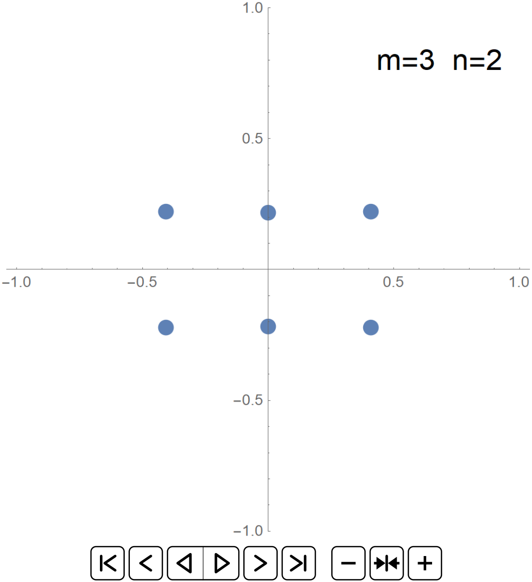
$$\psi''(z) = \left(E^2 V(z; \boldsymbol{\alpha}, \boldsymbol{\beta}, \nu) - \frac{1}{4z^2} \right) \psi(z),$$

$$V = z^2 + 2\boldsymbol{\alpha} z + \boldsymbol{\alpha}^2 - 1 - \boldsymbol{\beta} z^{-1} + \frac{\nu^2}{4} z^{-2},$$

satisfies **apparent singularity** and **quantisation conditions**.

Next step: Apply complex WKB approach as $E \to \infty$ to approximate ratios of solutions and solve apparent singularity and quantisation conditions .

The elliptic region, $\nu = \frac{1}{4}$, $\frac{m}{n} = \frac{3}{2}$



Complex WKB Approach

As $E \to \infty$ solutions of anharmonic oscillator are (locally) well-approximated by WKB functions

$$\psi = V^{-\frac{1}{4}} e^{\pm E \int^{z} \sqrt{V(\mu)} d\mu},$$

$$V = z^2 + 2\boldsymbol{\alpha} z + \boldsymbol{\alpha}^2 - 1 - \boldsymbol{\beta} z^{-1} + \frac{\nu^2}{4} z^{-2}.$$

Here, the term $-\frac{1}{4z^2}$ is neglected to get the correct approximation near the Fuchsian singularity (Langer modification).

It turns out that the region, where the roots condensate asymptotically, is determined by the Stokes geometry underlying the problem.

Stokes Geometry

Consider potential

$$V(z) = z^2 + 2\boldsymbol{\alpha}\, z + \boldsymbol{\alpha}^2 - 1 - \boldsymbol{\beta}\, z^{-1} + \frac{\nu^2}{4} z^{-2}$$

- **Turning points** are the zeros of $V(z)$
- **Stokes lines** are level sets $\Re \int_{z^*}^{z} \sqrt{V(\mu)}\, d\mu = 0$ in $\mathbb{CP}^1$, where z^* any turning point.
- **Stokes complex** $\mathcal{C} = \mathcal{C}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \subseteq \mathbb{CP}^1$ of $V(z)$ is the union of all its Stokes lines and zeros (decorated with some points at infinity).

Geometric Condition on Potential

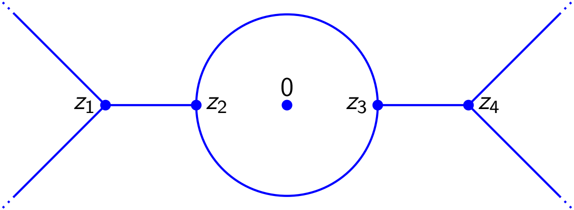


Figure: Stokes complex $\mathcal{C}(\alpha, \beta)$ with $(\alpha, \beta) = (0, 0)$, where $z_{1,2,3,4}$ are the zeros of $V = z^2 - 1 + \frac{\nu^2}{4} z^{-2}$.

To realise the apparent singularity and quantisation condition, we have to impose the following.

Geometric Condition on potential $V(z; \alpha, \beta, \nu)$

The Stokes complex $\mathcal{C}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of $V(\lambda)$ is homeomorphic to the Stokes complex $\mathcal{C}(0, 0)$.

Elliptic Region

Definition (Elliptic Region)

Let R be region where geometric condition on Stokes complex is satisfied,

$$R = \{(\alpha, \beta) \in \mathbb{C}^2 : \mathcal{C}(\alpha, \beta) \cong \mathcal{C}(0, 0)\}.$$

Denote its closure by $K = \overline{R}$ and let $K_a = K_a(\nu)$ be the projection of K onto α-plane. We call K_a the **elliptic region**.

Theorem (Elliptic Region, part 1, PR and Masoero 2021)

As $E \to \infty$, roots of $H_{m,n}(E^{\frac{1}{2}} z)$ **densely fill** *up elliptic region K_a.*

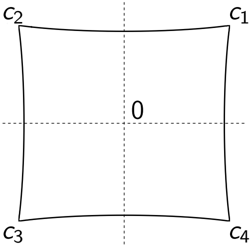
Elliptic Region, Corners

Theorem (Elliptic Region, part 2)

The elliptic region K_a is a compact quadrilateral domain whose boundary is a Jordan curve composed of four analytic pieces (edges), meeting at four corners $c_{1,2,3,4}$, as in figure.
The corner c_k is the unique solution of

$$\alpha^8 - 6(3\nu^2+1)\alpha^4 + 8(1-9\nu^2)\alpha^2 - 3(9\nu^4+6\nu^2+1) = 0$$

in k-th quadrant of complex α-plane. (Remaining four roots are purely real or imaginary)



Elliptic Region, Boundary Parametrisation

Theorem (Elliptic Region, part 3)

Cut α-plane along diagonals $[c_1, c_3]$ and $[c_2, c_4]$. Then

$$\psi(\alpha) = \tfrac{1}{2}\Re\left[\alpha y + \tfrac{1}{2}(1-\nu)\log(p_1) - \log(p_2) + \nu\log(x^{-2}p_3)\right],$$

$$p_1 = 1 - 2x\alpha - 2x^2, \quad p_2 = 2x + \alpha + y, \quad p_3 = x(\alpha^2 + 5x\alpha + 4x^2 - 1) + \frac{1}{2}\nu y,$$

is a **univalued harmonic function** *on this cut plane.*

Here $x = x(\alpha)$ and $y = y(\alpha)$ are the unique algebraic functions which solve

$$3x^4 + 4\alpha x^3 + (\alpha^2 - 1)x^2 - \frac{\nu^2}{4} = 0, \qquad x(\alpha) \sim \frac{\nu}{2}\alpha^{-1} \quad (\alpha \to \infty),$$

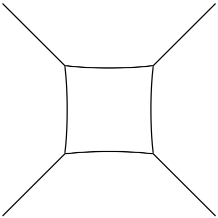
$$y^2 = \alpha^2 + 6x\alpha + 6x^3 - 1, \qquad y(\alpha) \sim \alpha \quad (\alpha \to \infty),$$

on the same cut plane.

Elliptic Region, Boundary Parametrisation

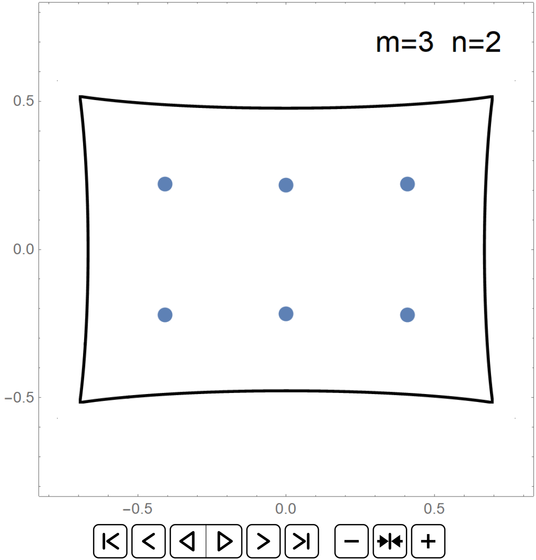
Theorem (Elliptic Region, part 4)

The level set $\{\psi(\alpha) = 0\}$ consists of **boundary elliptic region** ∂K_a *plus four additional lines which emanate from corners and go to infinity, see figure.*



Buckingham (2018) obtained different parametrisation elliptic region via Riemann-Hilbert approach to certain orthogonal polynomials and proved: asymptotically there are **no roots outside elliptic region**.

The Elliptic Region, $\nu = \frac{1}{4}$, $\frac{m}{n} = \frac{3}{2}$



WKB approach to quantisation condition

Based on the before mentioned Nevanlinna theory (Belyi functions characterisation), we simplify the asymptotic analysis by studying ratios of solutions of

$$\psi''(z) = \left(E^2 V(z; \boldsymbol{\alpha}, \boldsymbol{\beta}, \nu) - \frac{1}{4z^2}\right)\psi(z).$$

Let $f = \psi_*/\psi_0$ be a ratio of solutions defined on the cut plane $\mathbb{C} \setminus (-\infty, 0]$, where

- $\psi_0(z) \to 0$ as $z \to +\infty$,
- $\psi_*(z) \to 0$ as $z \to 0$.

The following function

$$U(\boldsymbol{\alpha}, \boldsymbol{\beta}, E) = \frac{\lim\limits_{z\to+i\infty} f(z)}{\lim\limits_{z\to-i\infty} f(z)},$$

is well-defined and meromorphic in the ϵ-neighbourhood of any compact subset of

$$R = \{(\alpha, \beta) \in \mathbb{C}^2 : \mathcal{C}(\alpha, \beta) \cong \mathcal{C}(0, 0)\},$$

for $E \geq E_0$, for some $\epsilon, E_0 > 0$.

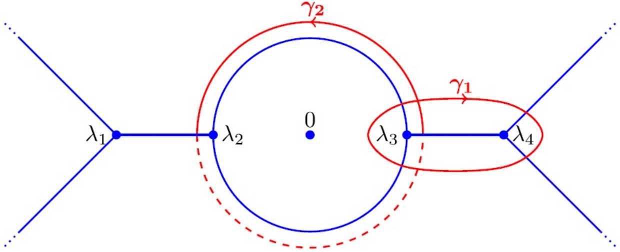
WKB approximation

Take a compact subset $K' \subset R$, then there exist $\epsilon, E_0, C > 0$, such that, for all $E \geq E_0$ and all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in an ϵ-neighbourhood of K',

- $U(\boldsymbol{\alpha}, \boldsymbol{\beta}, E) = 1$ if and only if the quantisation condition is satisfied.
- The following estimate holds,

$$\left| U - \exp\left(\int_{\gamma_1} \sqrt{V(\mu)} d\mu \right) \right| < \frac{C}{E} \exp\left(E \Re \int_{\gamma_1} \sqrt{V(\mu)} d\mu \right),$$

with branch determined by $\sqrt{V(\mu)} \sim +\frac{\nu}{2\mu}$ as $z \to 0$.



Anti-Stokes complex

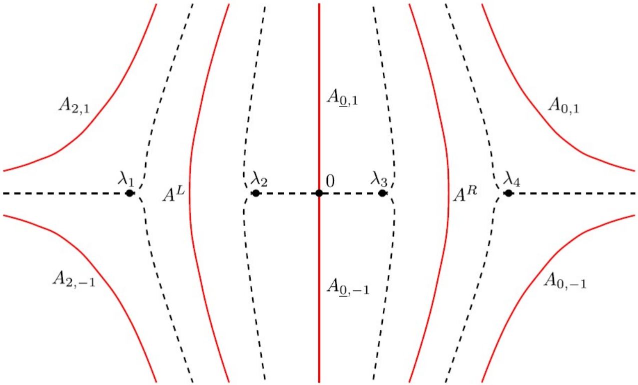


Figure: Anti-Stokes complex in dashed black

Two Complete Elliptic Integrals

Let $\omega := \frac{p}{z} dz$ be pull-back of $\sqrt{V} dz$ on elliptic curve

$$p^2 = z^4 + 2\alpha z^3 + (\alpha^2 - 1)z^2 - \beta z + \frac{\nu^2}{4}.$$

Assume $V(z; \alpha, \beta, \nu)$ satisfies the **geometric condition**, then we define the complete elliptic integrals

$$s_1(\alpha, \beta) = \int_{\gamma_1} \omega + \frac{i\pi(1 - \nu)}{2},$$
$$s_2(\alpha, \beta) = \int_{\gamma_2} \omega.$$

WKB Result (heuristically)

As E grows large,

- The **quantisation condition** is asymptotically equivalent to quantisation

$$s_1(\alpha, \beta) = i\frac{\pi j}{E}, \quad j \in I_m := \{-m+1, -m+3, \ldots, +m-1\}. \qquad (2)$$

- The **apparent singularity condition** is asymptotically equivalent to quantisation

$$s_2(\alpha, \beta) = i\frac{\pi k}{E}, \quad k \in I_n := \{-n+1, -n+3, \ldots, +n-1\}. \qquad (3)$$

Note: (2) and (3) are classical Bohr-Sommerfeld quantisation conditions.

Accounting: $\#(I_m \times I_n) = m \times n = \deg H_{m,n}$.

Asymptotic distribution within K_a

Recall Bohr-Sommerfeld quantisation conditions

$$s_1(\boldsymbol{\alpha}, \boldsymbol{\beta}) = i\frac{\pi j}{E}, \quad j \in \{-m+1, -m+3, \dots, +m-1\},$$
$$s_2(\boldsymbol{\alpha}, \boldsymbol{\beta}) = i\frac{\pi k}{E}, \quad k \in \{-n+1, -n+3, \dots, +n-1\}.$$

We may eliminate $\boldsymbol{\beta}$ by imposing $\Re s_{1,2}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 0$.
Then equations

$$\Im s_1(\boldsymbol{\alpha}) = \frac{\pi j}{E}, \quad j \in \{-m+1, -m+3, \dots, +m-1\}$$

define m 'vertical' grid lines within K_a.
Similarly equations

$$\Im s_2(\boldsymbol{\alpha}) = \frac{\pi k}{E}, \quad k \in \{-n+1, -n+3, \dots, +n-1\}$$

define n 'horizontal' grid lines within K_a.

Deformed rectangular lattice

Example $(m, n) = (4, 3)$:

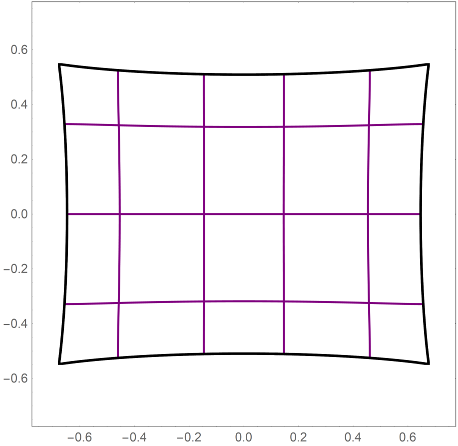


Figure: **deformed rectangular lattice** within elliptic region K_a

Asymptotic distribution within K_a

Theorem (Asymptotic Distribution of Bulk, heuristically)

In the large E limit, the bulk of the (rescaled) roots organise themselves within elliptic region K_a along the **vertices** *of* **deformed rectangular lattice** *defined by Bohr-Sommerfeld quantisation conditions.*

Example $(m, n) = (4, 3)$:

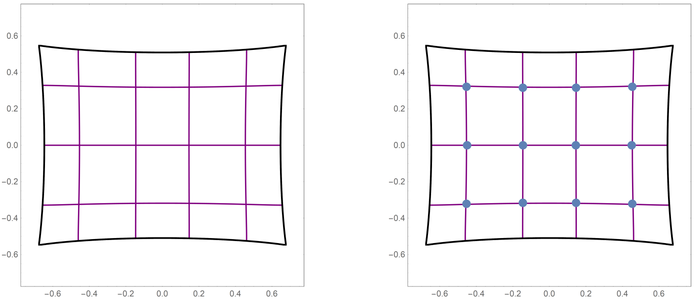


Figure: In both figures **deformed rectangular lattice** with on the right true locations of **roots** **superimposed**.

Asymptotic Distribution

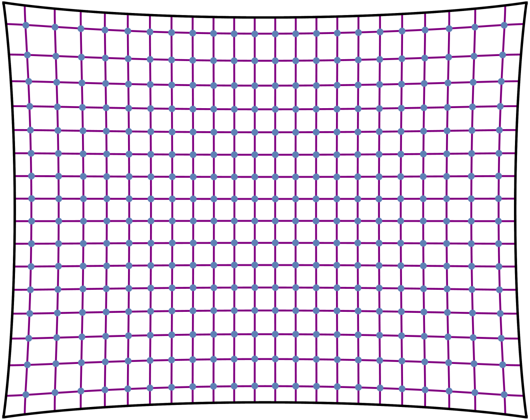
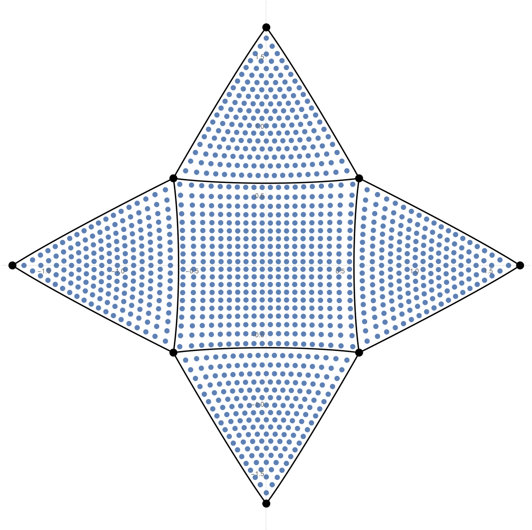


Figure: Asymptotic prediction are vertices of purple lattice, true location roots $H_{m,n}(z)$ in blue, with $(m,n) = (22,16)$.

Comments on literature

- Buckingham (2018) proved that asymptotically there are no zeros outside the elliptic region. Method: Riemann-Hilbert approach to associated orthogonal polynomials.
- PR and Masoero (2018,2021), proved that zeros densely fill up elliptic region, organising themselves along a deformed rectangular lattice.
- Buckingham and Miller (2020) computed asymptotic behaviour of rational solutions on the in and outside of elliptic region, by Riemann-Hilbert approach to the linear problem.

Thanks for your attention



$E \to \infty, \ n = \mathcal{O}(1).$

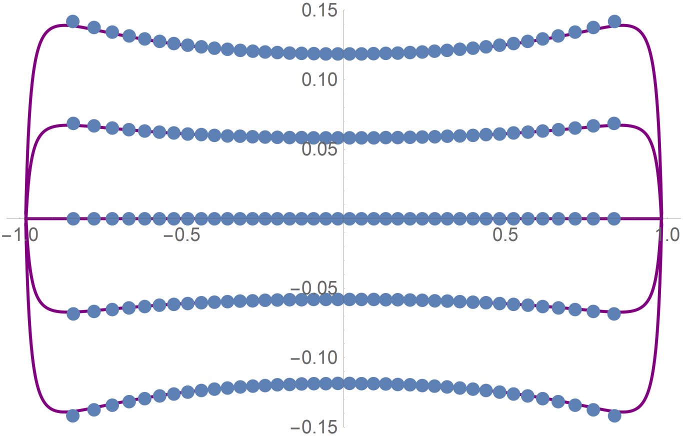


Figure: $(m, n) = (40, 5)$

Explicit formula $s_{1,2}$

$$s_1 = +\frac{2i}{\sqrt{(\lambda_4 - z_1)(z_3 - z_2)}} F(z_1, z_2, z_3, z_4) + \frac{1}{2} i\pi(1-\nu),$$
$$s_2 = -\frac{2}{\sqrt{(z_4 - z_3)(z_2 - z_1)}} F(z_4, z_1, z_2, z_3) + i\pi\nu,$$

with

$$\begin{aligned} F(z_1, z_2, z_3, z_4) = &- \tfrac{1}{4}(z_4 - z_2)(z_3 - z_2)(3z_1 - z_2 + z_3 + z_4)\mathcal{K}(m) \\ &+ \tfrac{1}{4}(z_4 - z_1)(z_3 - z_2)(z_1 + z_2 + z_3 + z_4)\mathcal{E}(m) \\ &+ (z_4 - z_2)\Pi(n_1, m) + 2z_1 z_3 (z_4 - z_2)\Pi(n_2, m), \end{aligned}$$

where

$$m = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}, \quad n_1 = -\frac{z_4 - z_3}{z_3 - z_2}, \quad n_2 = -\frac{(z_4 - z_3)z_2}{(z_3 - z_2)z_4},$$

and

$$z^4 + 2\alpha z^3 + (\alpha^2 - 1)z^2 - \beta z + \frac{\nu^2}{4} = (z - z_1)(z - z_2)(z - z_3)(z - z_4).$$