On partial-rogue wave solutions to the Sasa-Satsuma equation and generalised Okamoto polynomials

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Joint work with Alexander Stokes (University of Warsaw/Waseda University)

Sasa-Satsuma equation

The Sasa-Satsuma equation,

$$u_t = u_{xxx} + 6|u|^2 u_x + 3u\left(|u|^2\right)_x,$$

where $u(x,t) \in \mathbb{C}$, $(x,t) \in \mathbb{R}^2$, is an integrable extension of the nonlinear Schrödinger equation.

- Seems to have first appeared in [Kodama-Hasegawa,'87].
- Was shown to be **integrable** by [Sasa-Satsuma, '91].
- Has applications in **nonlinear optics** [Mihalache et al. '97], [Solli et al. '07], [Sun, '21]...
- It admits **rogue waves** [Bandelow-Akhmediev, '12], [Chen, '13],...
- It admits **partial-rogue wave** [Ohta-J.Zhang '13], [Zhao et al. '16], [B.Yang-J.Yang '23].

Rational Solutions

Bo Yang and Jianke Yang ('23) showed that the Sasa-Satsuma equation admits **rational solutions** on a nonzero background,

$$u_{M,N}(x,t) = R_{M,N}(x,t)u_{\mathrm{bg}}(x,t), \qquad u_{\mathrm{bg}}(x,t) := e^{i[\alpha(x+6t)-\alpha^3 t]},$$

with wavenumber parameter $\alpha = \frac{1}{2}$ and $R_{M,N}$ rational in $\{x,t\}$, indexed by $M, N \in \mathbb{Z}_{\geq 0}$, with $M+N$ free real parameters.

Some examples:

- $R_{0,0}(x,t) = 1$
- $R_{0,1}(x,t) = 1 + 12\dfrac{4+(33t+4x)i}{16+3267t^2+792tx+48x^2} = 1 + \dfrac{1+i\hat{x}}{\frac{1}{3}+\hat{x}^2}$,

  where $\hat{x} = x + \frac{33}{4}t$.
- $R_{1,0}(x,t) =$

  $$1 + \frac{3\left(9\sqrt{3}t+9\hat{x}^2+2\right) - 3\left(9t\left(2\sqrt{3}\hat{x}+1\right)+\left(\sqrt{3}-2\hat{x}\right)\left(3\hat{x}^2-2\right)\right)i}{243t^2-54t\hat{x}\left(\sqrt{3}\hat{x}-2\right)+3\hat{x}^2\left(3\hat{x}^2-4\sqrt{3}\hat{x}+8\right)+4}$$

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- $R_{1,0}(x,t) = 1 + \dfrac{3\left(9\sqrt{3}t+9\hat{x}^2+2\right)-3\left(9t\left(2\sqrt{3}\hat{x}+1\right)+\left(\sqrt{3}-2\hat{x}\right)\left(3\hat{x}^2-2\right)\right)i}{243t^2-54t\hat{x}\left(\sqrt{3}\hat{x}-2\right)+3\hat{x}^2\left(3\hat{x}^2-4\sqrt{3}\hat{x}+8\right)+4}$

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$R_{2,2}(x,t)$

In[415]:= Style[$R_{2,2}$[x, t] - 1 // Together, FontSize → 8]

Out[415]=

$$
\begin{aligned}
&\Big(24\Big(-20132659200+6710886400\,i\sqrt{3}+166094438400\,i\,t-468084326400\sqrt{3}\,t-30168789811200\,t^{2}-17632434585600\,i\sqrt{3}\,t^{2}-672911799091200\,i\,t^{3}-412009522790400\sqrt{3}\,t^{3}+3583868770713600\,t^{4}-\\
&2920510527897600\,i\sqrt{3}\,t^{4}-44160215557570560\,i\,t^{5}+59297572821565440\sqrt{3}\,t^{5}+272034510115737600\,t^{6}-179224251785441280\,i\sqrt{3}\,t^{6}-209453669085450240\,i\,t^{7}-879699132681431040\sqrt{3}\,t^{7}-\\
&10255656245043021120\,t^{8}+4708167625381239360\,i\sqrt{3}\,t^{8}+37874164723751236320\,i\,t^{9}-12248581425295057920\sqrt{3}\,t^{9}-15631295123608966494\,t^{10}+37104589434829364910\,i\sqrt{3}\,t^{10}+\\
&36845195648506849593\,i\,t^{11}+20132659200\,i\,x-20132659200\sqrt{3}\,x+1087163596800\,t\,x-2023332249600\,i\sqrt{3}\,t\,x-144830575411200\,i\,t^{2}x-52829356032000\sqrt{3}\,t^{2}x+1012463566848000\,t^{3}x-\\
&976799504793600\,i\sqrt{3}\,t^{3}x-22057509194956800\,i\,t^{4}x+24529230308966400\sqrt{3}\,t^{4}x+142419499433164800\,t^{5}x-114429606592020480\,i\sqrt{3}\,t^{5}x-233393782950789120\,i\,t^{6}x-662112911066112000\sqrt{3}\,t^{6}x-\\
&9119279434023475200\,t^{7}x+3915926655744153600\,i\sqrt{3}\,t^{7}x+38014708880453817600\,i\,t^{8}x-12825219187200182400\sqrt{3}\,t^{8}x-18947024392253292720\,t^{9}x+43334824908688988400\,i\sqrt{3}\,t^{9}x+\\
&49126927531342466124\,i\,t^{10}x-20132659200\,i\sqrt{3}\,x^{2}-11279322316800\,i\,t\,x^{2}-1041865113600\sqrt{3}\,t\,x^{2}+102125430374400\,t^{2}x^{2}-136965626265600\,i\sqrt{3}\,t^{2}x^{2}-4368747803443200\,i\,t^{3}x^{2}+\\
&3852578665267200\sqrt{3}\,t^{3}x^{2}+29109148095283200\,t^{4}x^{2}-29278668378931200\,i\sqrt{3}\,t^{4}x^{2}-91653494429122560\,i\,t^{5}x^{2}-213541931154309120\sqrt{3}\,t^{5}x^{2}-3537015701846077440\,t^{6}x^{2}+\\
&1419278098143375360\,i\sqrt{3}\,t^{6}x^{2}+16903752277175470080\,i\,t^{7}x^{2}-5957987659057336320\sqrt{3}\,t^{7}x^{2}-10334740577592705120\,t^{8}x^{2}+22742394488930556000\,i\sqrt{3}\,t^{8}x^{2}+29773895473540888560\,i\,t^{9}x^{2}-\\
&181193932800\,i\,x^{3}-20132659200\sqrt{3}\,x^{3}+3940968038400\,t\,x^{3}-8123527987200\,i\sqrt{3}\,t\,x^{3}-418668399820800\,i\,t^{2}x^{3}+280921374720000\sqrt{3}\,t^{2}x^{3}+2839732892467200\,t^{3}x^{3}-3822346174464000\,i\sqrt{3}\,t^{3}x^{3}-\\
&17746921193472000\,i\,t^{4}x^{3}-38280861912268800\sqrt{3}\,t^{4}x^{3}-781510532354703360\,t^{5}x^{3}+293022697517383680\,i\sqrt{3}\,t^{5}x^{3}+4369626510334525440\,i\,t^{6}x^{3}-1611467094604308480\sqrt{3}\,t^{6}x^{3}-\\
&3340522206898652160\,t^{7}x^{3}+7061854304627020800\,i\sqrt{3}\,t^{7}x^{3}+10826871081287595840\,i\,t^{8}x^{3}-115762790400\,i\sqrt{3}\,x^{4}-18532741939200\,i\,t\,x^{4}+9121982054400\sqrt{3}\,t\,x^{4}+122439676723200\,t^{2}x^{4}-\\
&265270040985600\,i\sqrt{3}\,t^{2}x^{4}-1862241170227200\,i\,t^{3}x^{4}-4122693166694400\sqrt{3}\,t^{3}x^{4}-107582403922329600\,t^{4}x^{4}+37742433457766400\,i\sqrt{3}\,t^{4}x^{4}+723493763747758080\,i\,t^{5}x^{4}-\\
&279608503724605440\sqrt{3}\,t^{5}x^{4}-708595619645168640\,t^{6}x^{4}+1436618824821734400\,i\sqrt{3}\,t^{6}x^{4}+2624696019706083840\,i\,t^{7}x^{4}-267261050880\,i\,x^{5}+95126814720\sqrt{3}\,x^{5}+866333491200\,t\,x^{5}-\\
&9040444784640\,i\sqrt{3}\,t\,x^{5}-104326936657920\,i\,t^{2}x^{5}-266955569233920\sqrt{3}\,t^{2}x^{5}-9448056603279360\,t^{3}x^{5}+3111754601594880\,i\sqrt{3}\,t^{3}x^{5}+79551003873116160\,i\,t^{4}x^{5}-32269453363445760\sqrt{3}\,t^{4}x^{5}-\\
&103068453766569984\,t^{5}x^{5}+200039062505103360\,i\sqrt{3}\,t^{5}x^{5}+445402960919820288\,i\,t^{6}x^{5}-56623104000\,x^{6}-110981283840\,i\sqrt{3}\,x^{6}-2697524674560\,i\,t\,x^{6}-9631589990400\sqrt{3}\,t\,x^{6}-\\
&516952660377600\,t^{2}x^{6}+160813154304000\,i\sqrt{3}\,t^{2}x^{6}+5807246789836800\,i\,t^{3}x^{6}-2476522576281600\sqrt{3}\,t^{3}x^{6}-10410954925916160\,t^{4}x^{6}+19304584530739200\,i\sqrt{3}\,t^{4}x^{6}+53988237687250944\,i\,t^{5}x^{6}-\\
&18119393280\,i\,x^{7}-149484994560\sqrt{3}\,x^{7}-16113802936320\,t\,x^{7}+4780122439680\,i\sqrt{3}\,t\,x^{7}+271335649443840\,i\,t^{2}x^{7}-121839613378560\sqrt{3}\,t^{2}x^{7}-721105103093760\,t^{3}x^{7}+1274680737792000\,i\sqrt{3}\,t^{3}x^{7}+\\
&4674306293268480\,i\,t^{4}x^{7}-219131412480\,x^{8}+62851645440\,i\sqrt{3}\,x^{8}+7361286635520\,i\,t\,x^{8}-3485718282240\sqrt{3}\,t\,x^{8}-32777504686080\,t^{2}x^{8}+55102154342400\,i\sqrt{3}\,t^{2}x^{8}+283291290501120\,i\,t^{3}x^{8}+\\
&88332042240\,i\,x^{9}-44166021120\sqrt{3}\,x^{9}-882895749120\,t\,x^{9}+1407791923200\,i\sqrt{3}\,t\,x^{9}+11446112747520\,i\,t^{2}x^{9}-10701766656\,x^{10}+16137584640\,i\sqrt{3}\,x^{10}+277481521152\,i\,t\,x^{10}+3057647616\,i\,x^{11}\Big)\Big)\Big/\\
&\Big(107374182400+1083902106009600\,t^{2}+9401383098777600\sqrt{3}\,t^{3}+39782365161062400\,t^{4}-351230319054028800\sqrt{3}\,t^{5}-259111772905144320\,t^{6}+17979443474830295040\sqrt{3}\,t^{7}+\\
&226559575854554342400\,t^{8}+533022978459209610240\sqrt{3}\,t^{9}+2205532065359022250752\,t^{10}+1518468669150585316560\sqrt{3}\,t^{11}+\\
&1215891456400726036569\,t^{12}+75376676044800\,t\,x+1759574281420800\sqrt{3}\,t^{2}x+14105539982131200\,t^{3}x-142156688051404800\sqrt{3}\,t^{4}x-\\
&234308824848138240\,t^{5}x+12185663545094307840\sqrt{3}\,t^{6}x+189738398029330022400\,t^{7}x+525496614966689034240\sqrt{3}\,t^{8}x+\\
&2503005567883347225600\,t^{9}x+1959663665713054847040\sqrt{3}\,t^{10}x+1768569391128328780464\,t^{11}x+1932735283200\,x^{2}+90596966400000\sqrt{3}\,t\,x^{2}+\\
&2125676622643200\,t^{2}x^{2}-19110501443174400\sqrt{3}\,t^{3}x^{2}-33737739809587200\,t^{4}x^{2}+3629613072879452160\sqrt{3}\,t^{5}x^{2}+69836058187955896320\,t^{6}x^{2}+\\
&229983518493652746240\sqrt{3}\,t^{7}x^{2}+1275992716224716328960\,t^{8}x^{2}+1148304508621411680000\sqrt{3}\,t^{9}x^{2}+1179046260752219186976\,t^{10}x^{2}+\\
&1610612736000\sqrt{3}\,x^{3}+132633958809600\,t\,x^{3}-925991593574400\sqrt{3}\,t^{2}x^{3}+4611544134451200\,t^{3}x^{3}+620798285080166400\sqrt{3}\,t^{4}x^{3}+14780266821868584960\,t^{5}x^{3}+\\
&58659530791338639360\sqrt{3}\,t^{6}x^{3}+384760848224101662720\,t^{7}x^{3}+403248752118480153600\sqrt{3}\,t^{8}x^{3}+476382327576654216960\,t^{9}x^{3}+3623878656000\,x^{4}+\\
&3080296857600\sqrt{3}\,t\,x^{4}+1478066857574400\,t^{2}x^{4}+66182324001177600\sqrt{3}\,t^{3}x^{4}+1970725389690470400\,t^{4}x^{4}+9612443404255887360\sqrt{3}\,t^{5}x^{4}+\\
&75994021842087444480\,t^{6}x^{4}+94286600818091827200\sqrt{3}\,t^{7}x^{4}+129922452975451150080\,t^{8}x^{4}+1207959552000\sqrt{3}\,x^{5}+124262799114240\,t\,x^{5}+\\
&4399710307614720\sqrt{3}\,t^{2}x^{5}+169782457820774400\,t^{3}x^{5}+1049899516160901120\sqrt{3}\,t^{4}x^{5}+10272286414591229952\,t^{5}x^{5}+15411187848906178560\sqrt{3}\,t^{6}x^{5}+\\
&25197081789178404864\,t^{7}x^{5}+3551401082880\,x^{6}+168129850245120\sqrt{3}\,t\,x^{6}+9241128389836800\,t^{2}x^{6}+76468770381496320\sqrt{3}\,t^{3}x^{6}+\\
&962347537583308800\,t^{4}x^{6}+1796633364358103040\sqrt{3}\,t^{5}x^{6}+3563223687358562304\,t^{6}x^{6}+2826625351680\sqrt{3}\,x^{7}+290761903964160\,t\,x^{7}+\\
&3583318394142720\sqrt{3}\,t^{2}x^{7}+61697763972218880\,t^{3}x^{7}+149371771355136000\sqrt{3}\,t^{4}x^{7}+370205058426863616\,t^{5}x^{7}+4049684398080\,x^{8}+98089335521280\sqrt{3}\,t\,x^{8}+\\
&2590622519132160\,t^{2}x^{8}+8678234574028800\sqrt{3}\,t^{3}x^{8}+28045837759610880\,t^{4}x^{8}+1195879956480\sqrt{3}\,x^{9}+64332905840640\,t\,x^{9}+335500384665600\sqrt{3}\,t^{2}x^{9}+\\
&1510886882672640\,t^{3}x^{9}+717527973888\,x^{10}+7766424944640\sqrt{3}\,t\,x^{10}+54941341188096\,t^{2}x^{10}+81537269760\sqrt{3}\,x^{11}+1210828455936\,t\,x^{11}+12230590464\,x^{12}\Big)
\end{aligned}
$$

Partial-rogue waves

Yang and Yang ('23)'s main motivation is the construction of **partial-rogue waves**, which they describe as

localised waves that 'come from nowhere but leave **with** a trace'.

With regards to the rational solutions $u_{M,N}(x,t) = R_{M,N}(x,t)u_{\text{bg}}(x,t)$, this means
$\lim_{t\to-\infty} R_{M,N}(x,t) = 1$ and,
$\lim_{x\to\pm\infty} R_{M,N}(x,t) = 1$, for bounded t.

Theorem (Yang and Yang ('23))

The rational solution $u_{M,N}(x,t)$ is a partial-rogue wave iff the following polynomial has no real roots,

$$Q_{M,N}^{[YY]} := \mathcal{W}\left[p_2, p_5, \ldots, p_{3M-1}, p_1, p_4, \ldots, p_{3N-2}\right].$$

Here entries $p_j(z)$ of Wronskian are Schur polynomials generated by

$$\exp(z\epsilon + \epsilon^2) = \sum_{j=0}^{\infty} p_j(z)\epsilon^j.$$

Partial-rogue wave continued

Theorem (Yang and Yang ('23), continued)

Furthermore, in such case, the solution asymptotically splits, as $t \to +\infty$, into $\rho_{M,N}$ solitons, where $\rho_{M,N}$ the number of imaginary roots of $Q_{M,N}^{[YY]}$.

Problem

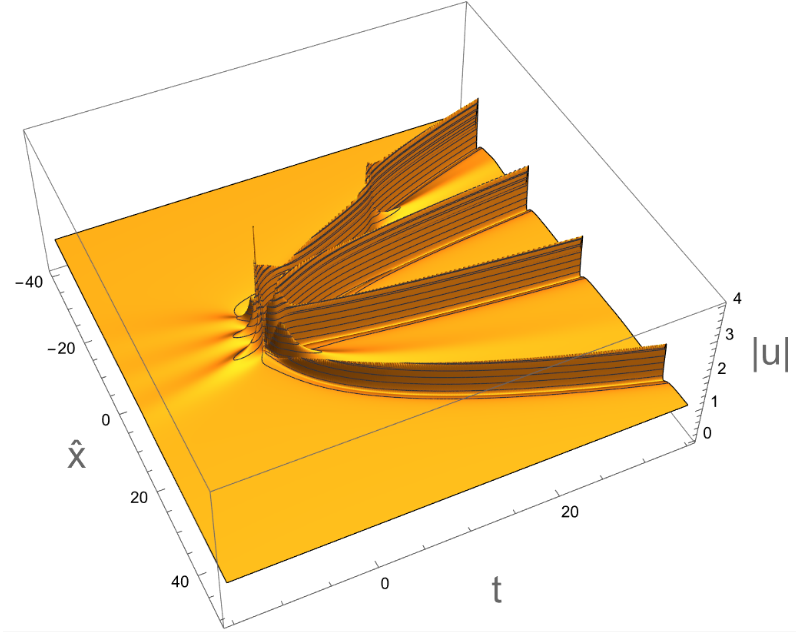
Determine the number of real and the number of imaginary roots of $Q_{M,N}^{[YY]}$, for $M, N \in \mathbb{Z}_{\geq 0}$.

Example $(M,N) = (3,0)$.
The polynomial

$$Q_{3,0}^{[YY]} = z^{12} + 28z^{10} + 260z^{8} + 1120z^{6} + 2800z^{4} + 11200z^{2} + 11200,$$

has no real roots and exactly $\rho_{3,0} = 4$ (purely) imaginary roots.

Numerical display of solution with $(M, N) = (3, 0)$



Generalised Okamoto polynomials

Noumi and Yamada (99'), based on previous work by Okamoto (86'), introduced the **generalised Okamoto polynomials** $Q_{m,n}$, $m,n \in \mathbb{Z}$, defined by recursive formulas

$$Q_{m,n-1}Q_{m,n+1} = \frac{9}{2}\left(Q''_{m,n}Q_{m,n} - (Q'_{m,n})^2\right) + \left(2t^2 + 3(m+2n+1)\right)Q_{m,n}^2,$$
$$Q_{m+1,n}Q_{m-1,n} = \frac{9}{2}\left(Q''_{m,n}Q_{m,n} - (Q'_{m,n})^2\right) + \left(2t^2 + 3(-2m+n-1)\right)Q_{m,n}^2,$$

with initial conditions $Q_{0,0} = Q_{-1,0} = Q_{0,-1} = 1, \quad Q_{-1,-1} = \sqrt{2}t.$

- The polynomials $Q_{M,N}^{[YY]}$, $M,N \in \mathbb{Z}_{\geq 0}$, form a subset of the generalised Okamoto polynomials,

  $$Q_{M,N}^{[YY]}(z) = \text{scalar} \times Q_{m,n}(t), \quad \tfrac{\sqrt{3}}{2}z = t, \quad (m,n) = (M-N, -M-1).$$

- Symmetry: $Q_{m,n}(it) = i^{\deg Q_{m,n}} Q_{n,m}(t)$

Problem

Determine the number of real roots of $Q_{m,n}$, $m,n \in \mathbb{Z}$.

Zero distributions of generalised Okamoto polynomials

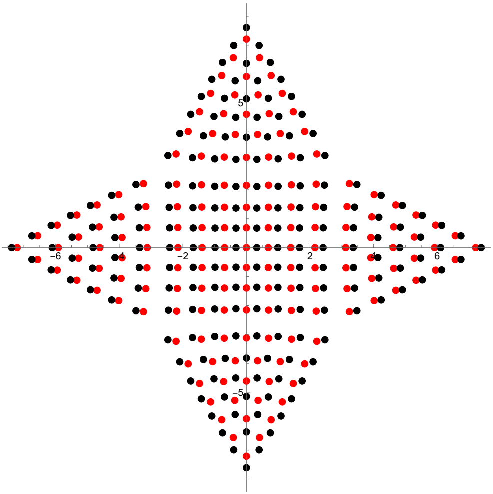


Figure: Roots of $Q_{7,7}$ (red) and $Q_{7,8}$ (black)

- Clarkson studied the locations of these zeroes numerically and observed beautiful patterns [Clarkson, 2003].
- Distributions have been studied in various large-parameter limits [Buckingham-Miller, 2022], [Masoero-R, 2024]
- Note that real roots seem to interlace.

Painlevé IV

The generalised Okamoto polynomials define rational solutions of the **fourth Painlevé equation**,

$$q'' = \frac{(q')^2}{2q} + \frac{3}{2}q^3 + 4\,t\,q^2 + 2\left(t^2 + a_2 - a_0\right)q - \frac{2a_1^2}{q}, \qquad ' = \frac{d}{dt},$$

where $(a_0, a_1, a_2) \in \mathbb{C}^3$ with $a_0 + a_1 + a_2 = 1$.

For $(m, n) \in \mathbb{Z}^2$ [Noumi-Yamada '99],

$$q_{m,n} = -\tfrac{2}{3}t + \frac{Q'_{m-1,n}}{Q_{m-1,n}} - \frac{Q'_{m,n}}{Q_{m,n}},$$

is a rational solution of Painlevé IV with parameter values

$$a_0 = \tfrac{1}{3} - m - n, \quad a_1 = \tfrac{1}{3} + n, \quad a_2 = \tfrac{1}{3} + m.$$

The fourth Painlevé equation is one of the six integrable second order nonlinear ODEs found around 1900 by Painlevé, Fuchs, Gambier and Picard without movable branch-points.

Apparent singularities

Zeros of generalised Okamoto polynomials correspond to poles of rational solutions

$$q_{m,n} = -\tfrac{2}{3}t + \frac{Q'_{m-1,n}}{Q_{m-1,n}} - \frac{Q'_{m,n}}{Q_{m,n}}.$$

- Any solution q of PIV is a meromorphic function $q : \mathbb{C} \to \mathbb{CP}^1$.
- Points $t_0 \in \mathbb{C}$, where $q(t_0) \in \{0, \infty\}$, are **singularities**.
- All these singularities are **apparent** as they can be resolved through a finite number of blow-ups - this is part of the construction of the **space of initial conditions** [Okamoto, 1979].
- Apparent singularities come in four types:
  - p_+ (plus pole) $\quad q(t) = \dfrac{+1}{t - t_0} + \mathcal{O}(1)$
  - p_- (minus pole) $\quad q(t) = \dfrac{-1}{t - t_0} + \mathcal{O}(1)$
  - z_+ (plus zero) $\quad q(t) = +2a_1(t - t_0) + \mathcal{O}((t - t_0)^2)$
  - z_- (minus zero) $\quad q(t) = -2a_1(t - t_0) + \mathcal{O}((t - t_0)^2)$

Singularity signatures of an Okamoto rational

To any real solution, we associate a **singularity signature**, which is a possibly infinite string of symbols from $\{p_+, p_-, z_+, z_-\}$.

Example:

$$\mathfrak{S}(q_{3,3}) = (p_-\, z_+\, z_-\, p_+)^1 (p_-\, z_+)^2\, p_-\, (z_+\, p_-)^2 (p_+\, z_-\, z_+\, p_-)^1$$

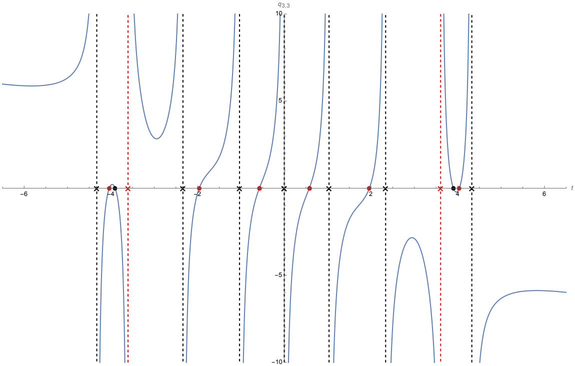


Figure: $q_{3,3}$, with $z_+ :$ ●, $z_- :$ ●, $p_+ : \times$, $p_- : \times$

Space of initial conditions for Painlevé IV

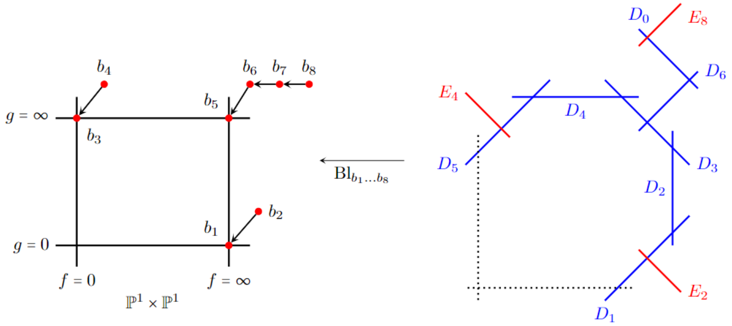


Figure: Okamoto's space of initial conditions $X_{t,a}$

By setting

$$f = q, \quad g = t + \frac{1}{2}q + \frac{a_1}{q} + \frac{q'}{2q},$$

Painlevé IV becomes a system of first order ODEs

$$f' = -2a_1 - f(2t + f - 2g),$$
$$g' = +2a_2 + g(2t + 2f - g),$$

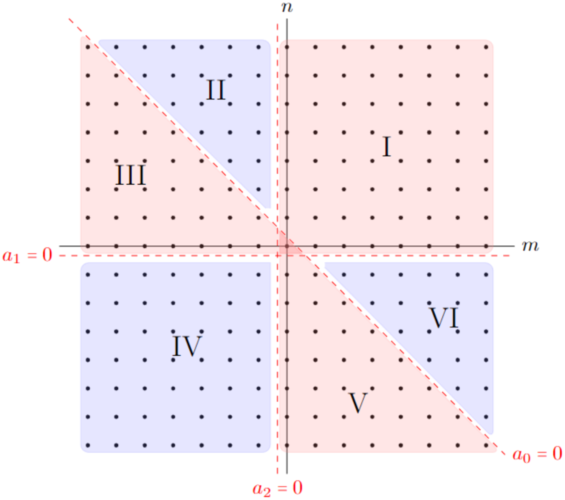
Method of attack

- Rational solutions parametrise **real paths** in the space of initial conditions.
- Apparent singularities correspond to paths **crossing** certain **exceptional curves**.
- The reality of the paths means that the order in which different exceptional curves can be crossed is greatly restricted by the geometry of Okamoto's space.
- The hierarchy of rational solutions $(q_{m,n})_{m,n\in\mathbb{Z}}$ is generated by Bäcklund transformations T_1, T_2 that act as isomorphisms between different Okamoto's spaces,

$$T_1 : q_{m,n} \mapsto q_{m,n+1}, \quad T_2 : q_{m,n} \mapsto q_{m+1,n},$$

  starting from the trivial solution $q_{0,0} = -\frac{2}{3}t$.
- By keeping track of how T_1 and T_2 act on exceptional curves plus geometric arguments and '**playing maze games**', we can inductively determine the singularity signatures of all these rational solutions.

Different regions in $\{(m, n) \in \mathbb{Z}^2\}$



Different regions are separated by lines $a_k = 0$, $0 \leq k \leq 2$, where some of the relevant exceptional curves undergo topological changes.

Maze game in region I for T_1 direction

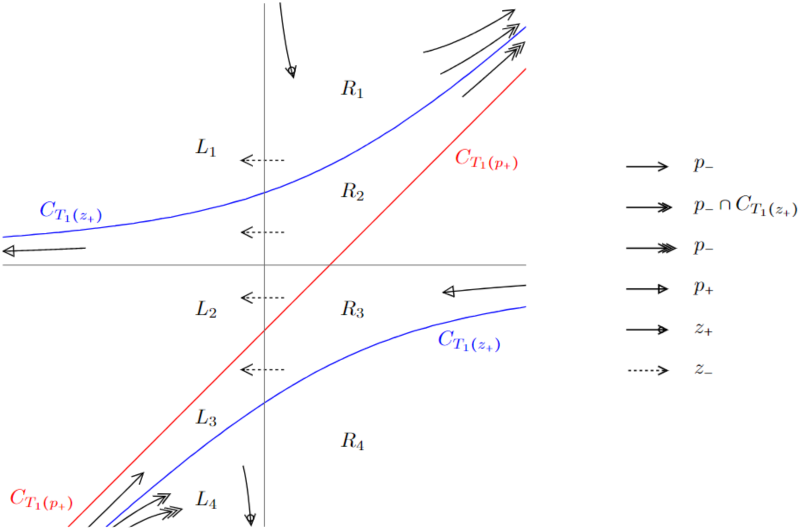


Figure: Relevant exceptional curves in (f, g) plane and path crossings

Results: singularity signatures

Theorem (Region I)

For $(m,n) \in \mathbb{Z}^2$ with $m \geq 0$, $n \geq 0$, the singularity signature of the generalised Okamoto rational $q_{m,n}(t)$ is as follows:

- $m = 2\mu$ *even,* $n = 2\nu$ *even*

$$\mathfrak{S}(q_{m,n}) = (p_- \, z_+ \, z_- \, p_+)^{\mu} \, (z_- \, p_+)^{\nu} \, \hat{z}_- \, (p_+ \, z_-)^{\nu} \, (p_+ \, z_- \, z_+, \, p_-)^{\mu}.$$

- $m = 2\mu$ *even,* $n = 2\nu + 1$ *odd*

$$\mathfrak{S}(q_{m,n}) = (p_- \, z_+ \, z_- \, p_+)^{\mu} \, (z_- \, p_+)^{\nu} \, z_- \, \hat{p}_+ \, z_- \, (p_+ \, z_-)^{\nu} \, (p_+ \, z_- \, z_+ \, p_-)^{\mu}.$$

- $m = 2\mu + 1$ *odd,* $n = 2\nu$ *even*

$$\mathfrak{S}(q_{m,n}) = (p_- \, z_+ \, z_- \, p_+)^{\mu} (p_- \, z_+)^{\nu} \, p_- \, \hat{z}_+ \, p_- (z_+ \, p_-)^{\nu} (p_+ \, z_- \, z_+ \, p_-)^{\mu}.$$

- $m = 2\mu + 1$ *odd,* $n = 2\nu + 1$ *odd*

$$\mathfrak{S}(q_{m,n}) = (p_- \, z_+ \, z_- \, p_+)^{\mu} (p_- \, z_+)^{\nu+1} \, \hat{p}_- \, (z_+ \, p_-)^{\nu+1} (p_+ \, z_- \, z_+ \, p_-)^{\mu}.$$

Results: number of real roots

region	m, n even	m even, n odd	m odd, n even	m odd, n odd
I	$\lvert m\rvert$	$\lvert m\rvert$	$\lvert m+n+1\rvert$	$\lvert m+n+1\rvert$
II	$\lvert m\rvert$	$\lvert m\rvert$	$\lvert n\rvert$	$\lvert n\rvert$
III	$\lvert n\rvert$	$\lvert m\rvert$	$\lvert n\rvert$	$\lvert m\rvert$
IV	$\lvert n\rvert$	$\lvert m+n+1\rvert$	$\lvert n\rvert$	$\lvert m+n+1\rvert$
V	$\lvert n\rvert$	$\lvert m+n+1\rvert$	$\lvert m+n+1\rvert$	$\lvert n\rvert$
VI	$\lvert m\rvert$	$\lvert m+n+1\rvert$	$\lvert m+n+1\rvert$	$\lvert m\rvert$

Table: Number of real roots of $Q_{m,n}(t)$ dependent on the parity of m and n as well as the region where the indices $(m, n) \in \mathbb{Z}^2$ lie.

Remark

[Hussin, Marquette, Zelaya, '22] derived entries in row IV from a conjecture/theorem? by [García-Ferrero, Gómez-Ullate '15].

Interlacing of real roots

Corollary (Interlacing of real roots)

(a) *Let* $(m,n) \in \mathbb{Z}^2$ *be in region* I *or region* II. *Then the real roots of* $Q_{m,n}$ *and* $Q_{m,n-1}$ *are interlaced.*

(b) *Let* $(m,n) \in \mathbb{Z}^2$ *be in region* III *or region* IV. *Then the real roots of* $Q_{m,n}$ *and* $Q_{m-1,n}$ *are interlaced.*

(c) *Let* $(m,n) \in \mathbb{Z}^2$ *be in region* V *or region* VI. *Then the real roots of* $Q_{m,n}$ *and* $Q_{m-1,n+1}$ *are interlaced.*

Results: real root free polynomials

Corollary

The polynomial $Q_{m,n}$ has no real roots if and only if

1. $m = 0$ *and* $n \geq 0$, *in which case the number of imaginary roots is*

$$\rho_{im}(Q_{0,n}) = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

2. $n = 0$ *and* $m \leq 0$, *in which case the number of imaginary roots is*

$$\rho_{im}(Q_{m,0}) \begin{cases} -m & \text{if } m \text{ is even,} \\ -m+1 & \text{if } m \text{ is odd,} \end{cases}$$

   or

3. $m \geq 0$, $n = -1 - m$, *in which case the number of imaginary roots is*

$$\rho_{im}(Q_{m,-1-m}) = \begin{cases} m & \text{if } m \text{ is even,} \\ m+1 & \text{if } m \text{ is odd.} \end{cases}$$

Partial-Rogue waves

Combining the previous results with the theorems in Yang and Yang ('23) yields the following corollary.

Corollary

For $M, N \in \mathbb{Z}_{\geq 0}$, the rational solution $u_{M,N}(x,t)$ is a partial-rogue wave if and only if $N = 0$ (and $M > 0$), in which case, as $t \to +\infty$, it asymptotically splits into M (resp. $M+1$) fundamental rational solitons if M is even (resp. odd).

The Sasa-Satsuma equation is invariant under

$$u \mapsto \hat{u}, \quad \hat{u}(x,t) = \overline{u(-x,-t)},$$

and correspondingly we have the following corollary.

Corollary

For $M, N \in \mathbb{Z}_{\geq 0}$, the rational solution $\widehat{u}_{M,N}(x,t)$ is a partial-rogue wave if and only if $M = N$, in which case, as $t \to +\infty$, it asymptotically splits into M (resp. $M+1$) fundamental rational solitons if M is even (resp. odd).

Thanks for your time!