MATH2021 - Differential Equations
Week 9, Lecture ~~1~~ 3

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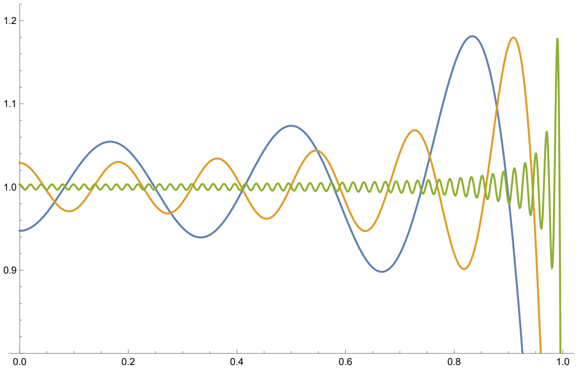


Past and present

- Last lecture was all about homogeneous linear 2nd order ODEs with constant coefficients:
  - the characteristic polynomial
  - roots of the characteristic polynomial and solutions
- Today is all about inhomogenous linear 2nd order ODEs:
  - existence and uniqueness
  - general and particular solutions
  - method of undetermined coefficients

Overview of the week

In **Lecture 9-1** we studied **homogeneous** linear second order ODEs:

$$y'' + p(x)y' + q(x)y = 0.$$

In **Lecture 9-2** we learnt how to solve the **constant coefficient** case

$$a\,y'' + b\,y' + c\,y = 0.$$

In today's lecture, we study **inhomogeneous** linear second order ODEs:

$$y'' + p(x)y' + q(x)y = f(x),$$

and we learn a method to solve the **constant coefficient** case

$$a\,y'' + b\,y' + c\,y = f(x),$$

for special choices of $f(x)$, called the **method of undetermined coefficients**.

Existence and uniqueness theorem

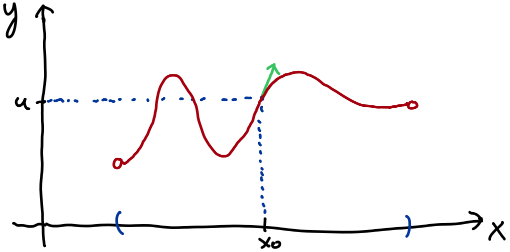
Theorem

Let $p(x)$, $q(x)$ and $f(x)$ be continuous functions on an open interval $\mathrm{I} \subseteq \mathbb{R}$. Take a point $x_0 \in \mathrm{I}$. Then, for any $u, v \in \mathbb{R}$, the **initial value problem**

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(x_0) = u, \quad y'(x_0) = v,$$

has a unique solution $y : \mathrm{I} \to \mathbb{R}$.

The proof of this theorem is technical and will be omitted.



The general solution

Theorem

Let $p(x)$, $q(x)$ and $f(x)$ be continuous functions on an open interval $\mathrm{I} \subseteq \mathbb{R}$. Let y_p be a **particular** solution of the **inhomogeneous** ODE

$$y'' + p(x)y' + q(x)y = f(x), \tag{1}$$

and $\{y_1, y_2\}$ be a fundamental set of solutions of the **homogeneous** ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Then the **general solution** of ODE (1) is given by

$$y = y_p + c_1\,y_1 + c_2\,y_2 \qquad (c_1, c_2 \text{ arbitrary constants}).$$

We prove this theorem using the linearity (superposition principle) of the differential operator

$$\mathcal{L}[y] = y'' + p(x)y' + q(x)y.$$

Proof of theorem

Proof: We have to check two things:

(i) $y = y_p + c_1 y_1 + c_2 y_2$ is a solution of ODE (1), for any $c_1, c_2 \in \mathbb{R}$.

(ii) Any solution y of ODE (1) can be written as $y = y_p + c_1 y_1 + c_2 y_2$, for some $c_1, c_2 \in \mathbb{R}$.

To check (i), we compute

$$\mathcal{L}[y_p + c_1 y_1 + c_2 y_2] = \mathcal{L}[y_p] + c_1 \mathcal{L}[y_1] + c_2 \mathcal{L}[y_2] = f(x) + c_1 \cdot 0 + c_2 \cdot 0 = f(x).$$

So $y = y_p + c_1 y_1 + c_2 y_2$ is indeed a solution of ODE (1).

To check (ii), take any solution y of ODE (1). Write $\tilde{y} = y - y_p$, then

$$\mathcal{L}[\tilde{y}] = \mathcal{L}[y - y_p] = \mathcal{L}[y] - \mathcal{L}[y_p] = f(x) - f(x) = 0.$$

So $\tilde{y}$ is a solution of the homogeneous ODE (2), which means $\tilde{y} = c_1 y_1 + c_2 y_2$ for some $c_1, c_2 \in \mathbb{R}$. Therefore

$$y = y_p + \tilde{y} = y_p + c_1 y_1 + c_2 y_2.$$

The theorem follows. $\square$

Example 1

Find the general solution of the inhomogeneous ODE

$$y'' + y = x \quad (*)$$

Step 1: find the general solution of the homogeneous ODE $y'' + y = 0$.

Characteristic polynomial: $P(\lambda) = \lambda^2 + 1$

The roots of $P(\lambda)$ are: $\lambda_1 = i$, $\lambda_2 = -i$.

The general solution is given by

$$y(x) = c_1 \cdot \cos x + c_2 \cdot \sin x \quad (c_1, c_2 \text{ arbitrary constants})$$

Step 2: find a particular solution to the inhomogeneous ODE.

Note that $y_p(x) = x$ is a particular solution.

Check: $y_p'' + y_p = 0 + x = x$ ✓

Step 3: Add together the solutions found in Steps 1 and 2.

The general solution of $(*)$ is

$$y = x + c_1 \cos x + c_2 \sin x \quad (c_1, c_2 \text{ arbitrary constants})$$

How to find a particular solution

The **method of undetermined coefficients** allows us to find **particular solutions** to ODEs of the form

$$a\,y'' + b\,y' + c\,y = f(x),$$

where $a, b, c \in \mathbb{R}$ are constants, with $a \neq 0$, and $f(x)$ takes one of a number of standard forms:

(1) $f(x) = P_n(x)$

(2) $f(x) = P_n(x)e^{\alpha x}$

(3) $f(x) = P_n(x)(c_1 \cos \beta x + c_2 \sin \beta x)$

(4) $f(x) = P_n(x)e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Here $P_n(x)$ is a polynomial of degree n in each case and $\alpha, \beta, c_1, c_2 \in \mathbb{R}$.

Note: (1) is a special case of both (2) and (3). In turn, (2) and (3) are special cases of (4).

Table for the method

$f(x)$	ansatz $y_p(x)$
c	A
$P_n(x)$	$A_0 + A_1 x + \ldots + A_n x^n$
$c\, e^{\alpha x}$	$A\, e^{\alpha x}$
$P_n(x)\, e^{\alpha x}$	$(A_0 + A_1 x + \ldots + A_n x^n) e^{\alpha x}$
$c_1 \cos \beta x + c_2 \sin \beta x$	$A \cos \beta x + B \sin \beta x$
$P_n(x)\,(c_1 \cos \beta x + c_2 \sin \beta x)$	$(A_0 + A_1 x + \ldots + A_n x^n)(A \cos \beta x + B \sin \beta x)$
$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$	$e^{\alpha x}(A \cos \beta x + B \sin \beta x)$
$P_n(x) e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$	$(A_0 + A_1 x + \ldots + A_n x^n) e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

Here $P_n(x)$ is a polynomial of degree n in each case and $c, \alpha, \beta, c_1, c_2 \in \mathbb{R}$.

Modification rule: If any of the terms in the **ansatz** after expanding, is a solution of the homogeneous ODE $a\, y'' + b\, y' + c\, y = 0$, multiply the ansatz by x. If the same is still true, multiply by x again.

Procedure of the method

To find a particular solution to the ODE

$$a\,y'' + b\,y' + c\,y = f(x), \tag{3}$$

the procedure of the method is as follows.

Step 1: Identify $f(x)$ as one of the forms in the left-hand side of the table. Take the corresponding ansatz on the right-hand side

$$y_p(x; \text{coefficients}),$$

appropriately modified if necessary. In the particular solution, there are one or more coefficients yet to be determined.

Step 2: Substitute the ansatz into the ODE (3) and determine the coefficients.

Example 2

Find the general solution of the inhomogeneous ODE

$$y'' - 3y' + 2y = e^{5x}.$$

Step 1: solve homogeneous ODE $y'' - 3y' + 2y = 0$

Characteristic polynomial: $\lambda^2 - 3\lambda + 2 =: P(\lambda)$.

Note that $P(\lambda) = (\lambda - 1)(\lambda - 2)$, so roots are $\lambda_1 = 1$, $\lambda_2 = 2$.

The general solution of the homogeneous ODE is

$$y(x) = c_1 e^{x} + c_2 e^{2x}.$$

Example 2 continued

Step 2: apply the method of undetermined coefficients to find a particular solution.

$f(x) = e^{5x} \Rightarrow$ ansatz $y_p(x) = A e^{5x}$

We substitute ansatz in ODE:

$$e^{5x} = y_p'' - 3y_p' + 2y_p = [A e^{5x}]'' - 3[A e^{5x}]' + 2Ae^{5x}$$

$$= 25Ae^{5x} - 15Ae^{5x} + 2Ae^{5x} = 12Ae^{5x}$$

So $1 = 12A \Rightarrow A = \frac{1}{12}$.

Therefore $y_p(x) = \frac{1}{12}e^{5x}$ is a particular solution.

Step 3: add together solutions found in steps 1 and 2.

$$y(x) = \frac{1}{12}e^{5x} + c_1 e^{x} + c_2 e^{2x}$$ (c_1, c_2 arbitrary constants)

is the general solution.

Example 3

Find the general solution of the inhomogeneous ODE

$$y'' + 2\,y' + 5\,y = x^2.$$

Step 1: solve $y'' + 2y' + 5y = 0$

Characteristic polynomial: $P(\lambda) = \lambda^2 + 2\lambda + 5$.

Note $P(\lambda) = (\lambda+1)^2 + 4$, so

$\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$ are the roots.

So the general solution is given by

$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$ (c_1, c_2 arbitrary constants).

Example 3 continued

Step 2: find particular solution using the method of undetermined coefficients.

$f(x) = x^2 \Rightarrow$ ansatz $y_p(x) = A + Bx + Cx^2$.

We substitute the ansatz into our ODE

$$x^2 = y_p'' + 2y_p' + 5y_p = 2C + 2(B + 2Cx) + 5(A + Bx + Cx^2)$$
$$= 2C + 2B + 5A + (4C + 5B)x + 5Cx^2$$

Comparing coefficients gives:

$5C = 1 \Rightarrow C = \frac{1}{5}$

$4C + 5B = 0 \Rightarrow B = -\frac{4}{5}C = -\frac{4}{25}$

$2C + 2B + 5A = 0 \Rightarrow A = -\frac{1}{5}(2C + 2B)$

So $A = -\frac{1}{5}\left(\frac{2}{5} - \frac{8}{25}\right) = -\frac{1}{5} \cdot \frac{2}{25} = -\frac{2}{125}$

So a particular solution is given by

$$y_p(x) = -\frac{2}{125} - \frac{4}{25}x + \frac{1}{5}x^2.$$

$$\Rightarrow y(x) = -\frac{2}{125} - \frac{4}{25}x + \frac{1}{5}x^2 + e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$ is the general solution

Example 4

Find a particular solution of the inhomogeneous ODE

$$y'' + y = 3\cos x.$$

Step 1: find general solution of $y'' + y = 0$.

Characteristic polynomial: $\lambda^2 + 1 = 0$.

Roots: $\lambda_1 = i$, $\lambda_2 = -i$.

Therefore, the general solution is given by

$y(x) = c_1 \cos x + c_2 \sin x$ (c_1, c_2 arbitrary constants).

Example 4 continued

Step 2: Apply method of undetermined coefficient to find particular solution.

$f(x) = 3\cos x$ implies the initial ansatz $A\cos x + B\sin x$.

But both terms in this ansatz are solutions to the homogeneous ODE.

We thus modify the ansatz: $y_p(x) = x\cdot(A\cos x + B\sin x)$.

Note: $y_p' = A\cos x + B\sin x + x(-A\sin x + B\cos x)$

$y_p'' = -2A\sin x + 2B\cos x - x(A\cos x + B\sin x)$

Substitution gives

$$3\cos x = y_p'' + y_p = -2A\sin x + 2B\cos x - x(A\cos x + B\sin x) + x(A\cos x + B\sin x)$$

$$= -2A\sin x + 2B\cos x$$

Comparing coefficients: $3 = 2B$, $0 = -2A$, so $A = 0$, $B = \frac{3}{2}$.

Hence $y_p(x) = \frac{3}{2}x\sin x$ is a particular solution.

Divide and conquer

If $y_{p,1}$ is a particular solution to

$$a\,y'' + b\,y' + c\,y = f_1(x),$$

and $y_{p,2}$ is a particular solution to

$$a\,y'' + b\,y' + c\,y = f_2(x),$$

then their sum $y_p = y_{p,1} + y_{p,2}$ is a particular solution to

$$a\,y'' + b\,y' + c\,y = f_1(x) + f_2(x).$$

This is another instance of the **superposition principle**. It allows us to divide a complicated problem into two simpler parts.

summary

In today's lecture, we considered inhomogeneous linear 2nd order ODEs.
We learnt about

- an **existence** and **uniqueness** theorem,
- general vs particular solutions,
- the **method of undetermined coefficients**.