MATH2021 - Differential Equations
Week 9, Lecture 2

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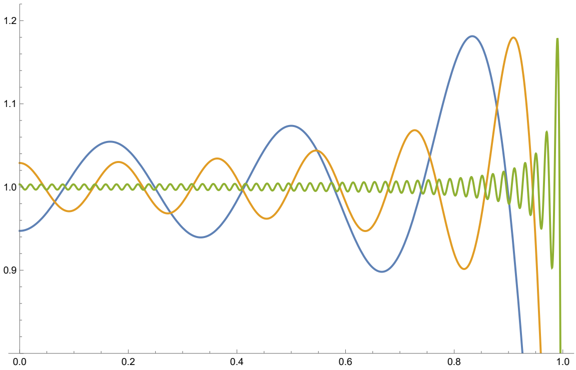
CONSULTATION HOUR: TODAY 1-2 PM



Past and present

- Last lecture:
  - differential operators
  - the superposition principle
  - fundamental set of solutions
  - Wronskians
- Today is all about homogeneous linear 2nd order ODEs with **constant coefficients**:
  - the characteristic polynomial
  - roots of the characteristic polynomial and solutions

Introduction

- Consider the ODE
$$y'' - y = 0$$
on $\mathrm{I} = \mathbb{R}$.

- It has solutions
$$y_1(x) = e^x, \qquad y_2(x) = e^{-x}.$$

- Their **Wronskian** equals
$$\mathcal{W}(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^x(-e^{-x}) - e^x e^{-x} = -1 - 1 = -2.$$

- The Wronskian is nonzero and thus $\{y_1, y_2\}$ is a **fundamental set of solutions**.

- This ODE is an example of a **homogeneous linear 2nd** order ODE with **constant coefficients**.

Hom. linear 2nd order ODE with constant coefficients

A **homogeneous linear 2nd** order ODE with **constant coefficients** can be written as

$$a\,y'' + b\,y' + c\,y = 0, \tag{1}$$

where $a, b, c \in \mathbb{R}$ are constants with $a \neq 0$.

Since all the coefficients are constant and thus continuous on $\mathbb{R}$, we look for solutions on $\mathrm{I} = \mathbb{R}$.

To solve the ODE, we try the **ansatz**

$$y(x) = e^{\lambda x}.$$

The characteristic polynomial

Substituting

$$y(x) = e^{\lambda x}, \qquad y'(x) = \lambda e^{\lambda x}, \qquad y''(x) = \lambda^2 e^{\lambda x}$$

into the left-hand side of (1) gives

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= a\,\lambda^2 e^{\lambda x} + b\,\lambda\, e^{\lambda x} + c\; e^{\lambda x} \\ &= (a\,\lambda^2 + b\,\lambda + c)e^{\lambda x}. \end{aligned}$$

For this to equal zero, necessarily $a\,\lambda^2 + b\,\lambda + c = 0$.

The polynomial

$$P(\lambda) = a\,\lambda^2 + b\,\lambda + c,$$

is known as the **characteristic polynomial**.

The **ansatz** $y(x) = e^{\lambda x}$ satisfies ODE (1) $\iff$
λ is a root of the characteristic polynomial $P(\lambda)$.

Roots of the characteristic polynomial

The roots of the characteristic polynomial

$$P(\lambda) = a\,\lambda^2 + b\,\lambda + c,$$

are given by

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4a\,c}}{2a}.$$

There are three distinct cases.

(1) $b^2 > 4a\,c \implies p(\lambda)$ has two distinct real roots,

(2) $b^2 = 4a\,c \implies p(\lambda)$ has a double root,

(3) $b^2 < 4a\,c \implies p(\lambda)$ has two distinct (non-real) complex roots.

Case 1

- Case (1): $b^2 > 4a\,c$.
- The characteristic polynomial has two distinct roots $\lambda_1, \lambda_2 \in \mathbb{R}$.
- Therefore,
$$y_1(x) = e^{\lambda_1 x}, \qquad y_2(x) = e^{\lambda_2 x},$$
are solutions of the ODE.
- Their Wronskian equals
$$\mathcal{W}(y_1, y_2)(x) = y_1 \cdot y_2' - y_1' \cdot y_2$$
$$= e^{\lambda_1 x} \cdot \lambda_2 e^{\lambda_2 x} - \lambda_1 e^{\lambda_1 x} \cdot e^{\lambda_2 x}$$
$$= (\lambda_2 - \lambda_1)\, e^{(\lambda_1 + \lambda_2) x}$$
- The Wronskian is nonzero on $\mathbb{R}$, so the two solutions are linearly independent.
- Therefore, the **general solution** of ODE (1) is
$$y(x) = c_1\, e^{\lambda_1 x} + c_2\, e^{\lambda_2 x}$$
where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Case 2

- Case (2): $b^2 = 4a\,c$.
- The characteristic polynomial has a double root $\boxed{\lambda = -\frac{b}{2a}} \in \mathbb{R}$.
- So $y_1(x) = e^{\lambda x}$ is a solution of the ODE.

claim: $y_2(x) = x\,e^{\lambda x}$ is another solution.

To check this claim, note that

$$y_2'(x) = e^{\lambda x} + x \cdot \lambda e^{\lambda x}$$

$$y_2''(x) = \lambda e^{\lambda x} + \lambda e^{\lambda x} + x \lambda^2 e^{\lambda x} = 2\lambda e^{\lambda \cdot x} + \lambda^2 x e^{\lambda \cdot x}$$

and therefore

$$\begin{aligned} a\,y_2'' + b\,y_2' + c\,y_2 &= a \cdot (2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}) + b\,(e^{\lambda x} + x \lambda e^{\lambda x}) + c \cdot x e^{\lambda x} \\ &= \left[2\lambda a + b + x\,(a\lambda^2 + b \cdot \lambda + c)\right] e^{\lambda \cdot x} \\ &= \left[2 \cdot \left(-\frac{b}{2a}\right) + b + x \cdot 0\right] e^{\lambda \cdot x} \\ &= \left[0 + x \cdot 0\right] e^{\lambda x} = 0 \end{aligned}$$

Case 2 continued

- Case (2): $b^2 = 4a\,c$.
- The characteristic polynomial has a double root $\lambda = -\frac{b}{2a} \in \mathbb{R}$.
- The ODE has solutions

$$y_1(x) = e^{\lambda x}, \qquad y_2(x) = x\,e^{\lambda x}.$$

- Their Wronskian equals

$$\begin{aligned} \mathcal{W}(y_1, y_2)(x) &= y_1 \cdot y_2' - y_1' \cdot y_2 \\ &= e^{\lambda x}(e^{\lambda \cdot x} + \lambda x e^{\lambda \cdot x}) - \lambda e^{\lambda \cdot x} \cdot x e^{\lambda \cdot x} \\ &= e^{2 \cdot \lambda \cdot x} \end{aligned}$$

- The Wronskian is nonzero on $\mathbb{R}$, so the two solutions are linearly independent.
- Therefore, the **general solution** of ODE (1) is

$$y(x) = e^{\lambda x}(c_1 + c_2\,x)$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Case 3

- Case (3): $b^2 < 4a\,c$.
- The characteristic polynomial has two complex roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4a\,c}}{2a} = \boldsymbol{\alpha} + \boldsymbol{\beta}\, i,$$
$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4a\,c}}{2a} = \boldsymbol{\alpha} - \boldsymbol{\beta}\, i,$$

  where

$$\boldsymbol{\alpha} = -\frac{b}{2a}, \qquad \boldsymbol{\beta} = \frac{\sqrt{4a\,c - b^2}}{2a}.$$

- The functions

$$y_1(x) = e^{\lambda_1 x}, \qquad y_2(x) = e^{\lambda_2 x},$$

  satisfy ODE (1), but they are generally **complex-valued**!

Case 3 continued

Any complex combination $d_1\, y_1(x) + d_2\, y_2(x)$, where $d_1, d_2 \in \mathbb{C}$, also satisfies ODE (1).

Using **Euler's formula**, $e^{iz} = \cos z + i \sin z$, we rewrite

$$
\begin{aligned}
d_1\, e^{\lambda_1 x} + d_2\, e^{\lambda_2 x} &= d_1 e^{(\alpha+\beta\, i)x} + d_2 e^{(\alpha-\beta\, i)x} \\
&= e^{\alpha\, x}\left(d_1\, e^{\beta\, i\, x} + d_2\, e^{-\beta\, i\, x}\right) \\
&= e^{\alpha\, x}\left(d_1(\cos\beta x + i\sin\beta x) + d_2(\cos\beta x - i\sin\beta x)\right) \\
&= e^{\alpha\, x}\left((d_1 + d_2)\cos\beta x + (d_1 - d_2)i\sin\beta x\right) \\
&= e^{\alpha\, x}\left(c_1\cos\beta x + c_2\sin\beta x\right),
\end{aligned}
$$

where $c_1 = d_1 + d_2$ and $c_2 = (d_1 - d_2)i$.

If we ensure that $c_1, c_2 \in \mathbb{R}$, then

$$y(x) = e^{\alpha\, x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$$

is a (real-valued) solution of ODE (1).

Case 3 continued

- Case (3): $b^2 < 4ac$.
- The characteristic polynomial has distinct complex roots

$$\lambda_1 = \alpha + \beta i, \qquad \lambda_2 = \alpha - \beta i,$$

which are complex conjugates of each other.
- The ODE has solutions

$$y_1(x) = e^{\alpha x} \cos \beta x, \qquad y_2(x) = e^{\alpha x} \sin \beta x.$$

- Their Wronskian equals

$$\mathcal{W}(y_1, y_2)(x) = \beta e^{2 \alpha x}. \qquad \text{(exercise)}$$

- The Wronskian is nonzero on $\mathbb{R}$, so the two solutions are linearly independent.
- Therefore, the **general solution** of ODE (1) is

$$y(x) = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Summary

Let $a, b, c \in \mathbb{R}$, with $a \neq 0$, and denote by

$$P(\lambda) = a\lambda^2 + b\lambda + c$$

the **characteristic polynomial** of the ODE

$$a\, y'' + b\, y' + c\, y = 0.$$

Then, either

(1) $P(\lambda)$ has two **distinct real** roots λ_1, λ_2, and the general solution of the ODE is

$$y(x) = c_1\, e^{\lambda_1 x} + c_2\, e^{\lambda_2 x}, \qquad (c_1, c_2 \in \mathbb{R}).$$

(2) $P(\lambda)$ has a **real double** root λ and the general solution of the ODE is

$$y(x) = e^{\lambda x}(c_1 + c_2\, x), \qquad (c_1, c_2 \in \mathbb{R}).$$

(3) $P(\lambda)$ has **distinct complex** roots

$$\lambda_1 = \boldsymbol{\alpha} + \boldsymbol{\beta}\, i, \qquad \lambda_2 = \boldsymbol{\alpha} - \boldsymbol{\beta}\, i,$$

and the general solution of the ODE is

$$y(x) = e^{\boldsymbol{\alpha}\, x}\left(c_1 \cos \boldsymbol{\beta} x + c_2 \sin \boldsymbol{\beta} x\right), \qquad (c_1, c_2 \in \mathbb{R}).$$

Example 1

Find the general solution to the ODE

$$y'' - 2y = 0.$$

The characteristic polynomial is given by

$P(\lambda) = \lambda^2 - 2.$

This polynomial has 2 real roots $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$.

The general solution of the ODE is

$$y(x) = c_1 e^{\sqrt{2}x} + c_2 \cdot e^{-\sqrt{2} \cdot x},$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Example 2

Find a fundamental set of solutions to the ODE

$$9\,y'' + 6\,y' + y = 0.$$

Characteristic polynomial: $P(\lambda) = 9\lambda^2 + 6\lambda + 1$

$P(\lambda) = (3\lambda + 1)^2$,

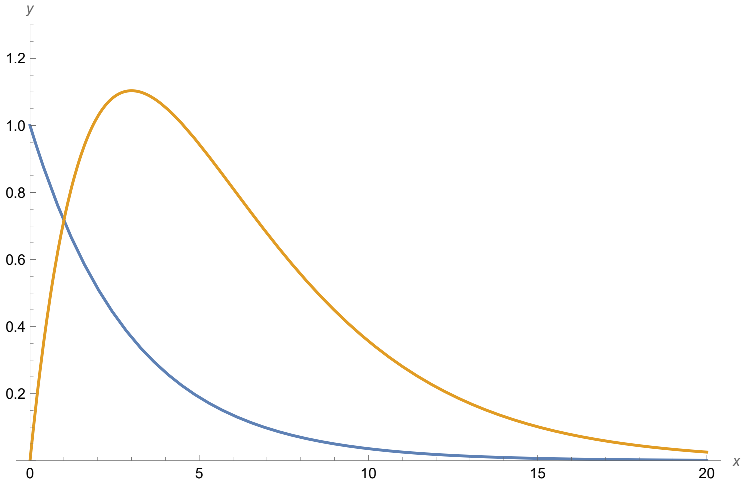
so $P(\lambda)$ has a double root at $\lambda = -\frac{1}{3}$.

Therefore, the general solution is given by

$$y(x) = e^{-\frac{1}{3}x}\left(c_1 + c_2 \cdot x\right).$$

So $\{y_1(x) = e^{-\frac{1}{3}x},\ y_2(x) = x \cdot e^{-\frac{1}{3}x}\}$ form a fundamental set of solutions to the ODE.

Example 2 plot



- $y_1(x) = e^{-x/3}$ in blue.
- $y_2(x) = x\, e^{-x/3}$ in orange.

Example 3

Find a fundamental set of solutions to the ODE

$$y'' + 2\,y' + 10\,y = 0.$$

Characteristic polynomial $P(\lambda) = \lambda^2 + 2\lambda + 10$

$P(\lambda) = (\lambda+1)^2 + 9$,

so its roots are $\lambda_1 = -1 + 3i$,

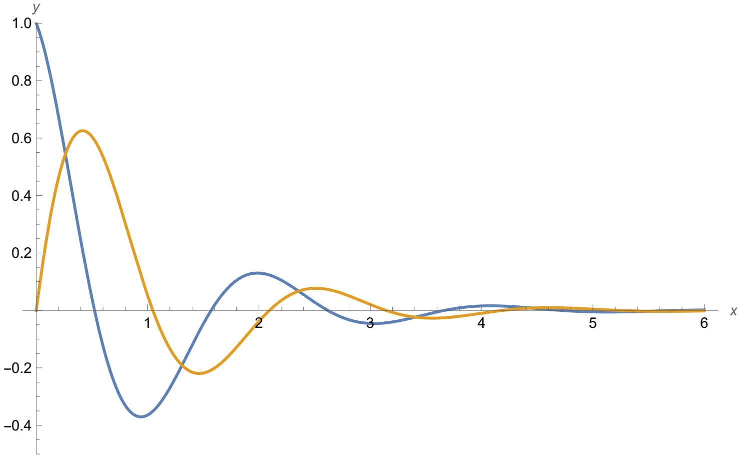
$\lambda_2 = -1 - 3i$.

(So $\alpha = -1$, $\beta = 3$)

A fundamental set of solutions is given by

$y_1(x) = e^{-x} \cdot \cos 3x$, $\quad y_2(x) = e^{-x} \sin 3x$.

Example 3 plot



- $y_1(x) = e^{-x} \cos x$ in blue.
- $y_2(x) = e^{-x} \sin x$ in orange.

After today's lecture, you should

- know what a **characteristic polynomial** is in the context of ODEs,
- know how to solve homogeneous linear 2nd order ODEs with **constant coefficients**.