MATH2021 - Differential Equations
Week 9, Lecture 1

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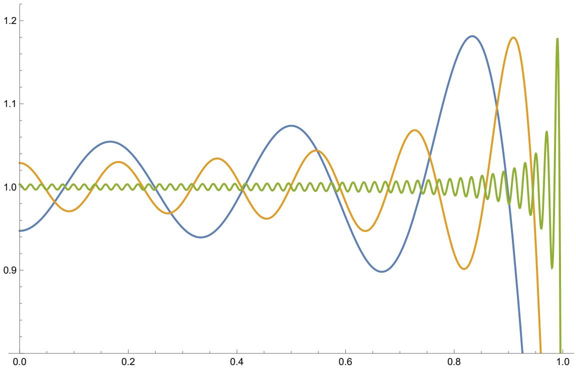


Past and present

- Last lecture:
  - Linear first order ODEs
  - method of variation of parameters
- Today is all about homogeneous linear 2nd order ODEs:
  - differential operators
  - existence and uniqueness
  - principle of superposition
  - Wronskians and Abel's formula

A differential operator

Let $p(x)$ be a continuous function on an open interval $\mathrm{I} \subseteq \mathbb{R}$.
Define the **differential operator**

$$\mathcal{L}[y] = y' + p(x)y.$$

It takes in a continuously differentiable function on I and outputs a continuous function.

The superposition principle

The operator $\mathcal{L}$ is **linear**, that is,

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2], \quad \mathcal{L}[c\,y] = c\,\mathcal{L}[y],$$

for any $c \in \mathbb{R}$ and continuously differentiable functions y_1, y_2 on I.

Check:

$$\begin{aligned}
\mathcal{L}[y_1+y_2] &= (y_1+y_2)' + p(x)(y_1+y_2) \\
&= y_1' + y_2' + p(x)\,y_1 + p(x)\,y_2 \\
&= (y_1' + p(x)y_1) + (y_2' + p(x)y_2) \\
&= \mathcal{L}[y_1] + \mathcal{L}[y_2] \quad \checkmark
\end{aligned}$$

$$\mathcal{L}[c \cdot y] = (cy)' + p(x)(cy) = c(y' + p(x)y) = c \cdot \mathcal{L}[y] \quad \checkmark$$

Structure of the general solution

Let $f(x)$ be a continuous function on I. The ODE

$$y' + p(x)y = f(x), \tag{1}$$

can be written as

$$\mathcal{L}[y] = f(x).$$

Given a particular solution y_p of (1) and any non-trivial solution y_{hom} to the homogeneous ODE

$$\mathcal{L}[y] = 0,$$

then the general solution of (1) is given by

$$y = y_p + c\, y_{\text{hom}} \qquad (c \text{ an arbitrary constant}).$$

This works because

$$\begin{aligned}\mathcal{L}[y_p + c\, y_{hom}] &= \mathcal{L}[y] + \mathcal{L}[c\, y_{hom}] \\ &= \mathcal{L}[y] + c \cdot \mathcal{L}[y_{hom}] \\ &= f(x) + c \cdot 0 \\ &= f(x)\end{aligned}$$

Linear second order ODEs

- A linear second order ODE can be written as

$$y'' + p(x)y' + q(x)y = f(x).$$

- We will assume that $p(x)$, $q(x)$ and $f(x)$ are continuous on an open interval $\mathrm{I} \subseteq \mathbb{R}$.

- A **solution** to the ODE is a twice continuously differentiable function $y : \mathrm{I} \to \mathbb{R}$ that satisfies the equation on I.

- $f(x)$ is usually referred to as the **forcing function**, as it often corresponds to the effect of a force when the ODE models a physical system.

- If $f(x) \equiv 0$, the ODE is **homogeneous**. Otherwise, it is **inhomogeneous**.

The superposition principle

Define the **differential operator**

$$\mathcal{L}[y] = y'' + p(x)y' + q(x)y.$$

It takes in a twice continuously differentiable function on I and outputs a continuous function.

The superposition principle

The operator $\mathcal{L}$ is **linear**, that is,

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2], \quad \mathcal{L}[c\,y] = c\,\mathcal{L}[y],$$

for any $c \in \mathbb{R}$ and twice continuously differentiable functions y_1, y_2 on I.

Check:
$$\begin{aligned}
\mathcal{L}[y_1+y_2] &= (y_1+y_2)'' + p(x)(y_1+y_2)' + q(x)(y_1+y_2) \\
&= y_1'' + y_2'' + p(x)y_1' + p(x)y_2' + q(x)y_1 + q(x)y_2 \\
&= (y_1'' + p(x)y_1' + q(x)y_1) + (y_2'' + p(x)y_2' + q(x)y_2) \\
&= \mathcal{L}[y_1] + \mathcal{L}[y_2] \quad \checkmark
\end{aligned}$$

Set of solutions

Let V_{hom} denote the **set of solutions** to the **homogeneous** linear ODE

$$y'' + p(x)y' + q(x)y = 0.$$

The set V_{hom} is a **vector space** over $\mathbb{R}$ under the usual scalar multiplication and addition of functions.

Proof: Note that y is a solution of the ODE $\iff$ $\mathcal{L}[y] = 0$.

Since $\mathcal{L}[0] = 0$, the zero function lies in V_{hom}.

Take two solutions y_1 and y_2 of the ODE. Then any linear combination $c_1 y_1 + c_2 y_2$, where $c_1, c_2 \in \mathbb{R}$, also defines a solution to the ODE.

Indeed, by the **superposition principle**,

$$\mathcal{L}[c_1 y_1 + c_2 y_2] = \mathcal{L}[c_1 y_1] + \mathcal{L}[c_2 y_2] = c_1 \mathcal{L}[y_1] + c_2 \mathcal{L}[y_2] = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

It follows that V_{hom} is a vector space. $\square$

What is the dimension of V_{hom}?

Existence and uniqueness theorem

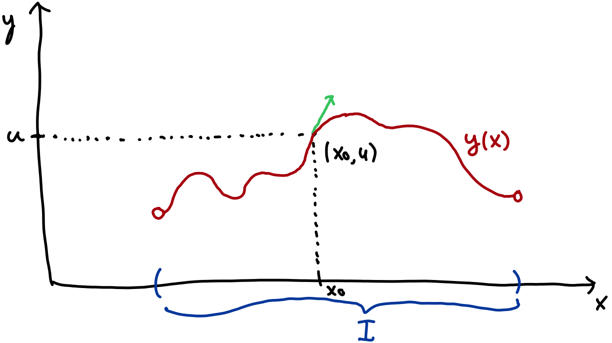
Theorem

Let $p(x)$ and $q(x)$ be continuous functions on an open interval $\mathrm{I} \subseteq \mathbb{R}$. Take any point $x_0 \in \mathrm{I}$. Then, for any $u, v \in \mathbb{R}$, the **initial value problem**

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = u, \quad y'(x_0) = v,$$

has a unique solution $y : \mathrm{I} \to \mathbb{R}$.

The proof of this theorem is technical and will be omitted.



Dimension of set of solutions 1/2

Corollary

The dimension of V_{hom} is two.

Proof of Corollary: Take an $x_0 \in \mathrm{I}$. By the theorem, there exist solutions y_1 and y_2 that satisfy

$$\begin{aligned} y_1(x_0) &= 1, & y_2(x_0) &= 0, \\ y_1'(x_0) &= 0, & y_2'(x_0) &= 1. \end{aligned}$$

We will prove that $\{y_1, y_2\}$ forms a basis of V_{hom}.

Step 1 - Linear independence: Suppose $c_1, c_2 \in \mathbb{R}$ are such that

$$c_1\, y_1(x) + c_2\, y_2(x) = 0. \tag{2}$$

Evaluating this equation at $x = x_0$ gives

$$c_1 \cdot 1 + c_2 \cdot 0 = 0 \implies c_1 = 0.$$

Evaluating the derivative of (2) at $x = x_0$ gives

$$c_1 \cdot 0 + c_2 \cdot 1 = 0 \implies c_2 = 0.$$

So $c_1 = c_2 = 0$. It follows that y_1 and y_2 are **linearly independent**.

Dimension of set of solutions 2/2

Step 2 - Completeness: Let $\tilde{y}$ be any other solution of the homogeneous ODE. Denote

$$\tilde{y}(x_0) = u,$$
$$\tilde{y}'(x_0) = v.$$

Define $\hat{y}(x) = u\, y_1(x) + v\, y_2(x)$, then

$$\hat{y}(x_0) = u\, y_1(x_0) + v\, y_2(x_0) = u \cdot 1 + v \cdot 0 = u,$$
$$\hat{y}'(x_0) = u\, y_1'(x_0) + v\, y_2'(x_0) = u \cdot 0 + v \cdot 1 = v.$$

So $\tilde{y}$ and $\hat{y}$ are solutions to the same initial value problem,

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = u, \quad y'(x_0) = v.$$

By the **uniqueness** part of the theorem, $\tilde{y} = \hat{y}$, so

$$\tilde{y}(x) = u\, y_1(x) + v\, y_2(x).$$

It follows that $\{y_1, y_2\}$ is a **basis** of V_{hom}. So the **dimension** of V_{hom} is two. $\square$

Fundamental set of solutions

We know that the set of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0,$$

forms a vector space of dimension two.
A **fundamental set of solutions** is by definition a set of solutions $\{y_1, y_2\}$ that is **linearly independent** (and thus forms a basis).

- $\{y_1, y_2\}$ forms a **fundamental set of solutions** if and only if any other solution y can be written as

$$y = c_1\, y_1 + c_2\, y_2$$

for some $c_1, c_2 \in \mathbb{R}$.

- In this case,

$$y = c_1\, y_1 + c_2\, y_2 \qquad (c_1, c_2 \text{ arbitrary constants}),$$

is the **general solution** of the ODE.

Example 1

Find a **fundamental set of solutions** for the ODE

$$y'' = 0, \qquad (*)$$

on the interval $\mathrm{I} = \mathbb{R}$.

INTEGRATE $(*)$: $\quad y' = c_1$

INTEGRATE AGAIN: $\quad y = c_1 \cdot x + c_2$

So $\quad y = c_1 \cdot x + c_2 \qquad (c_1, c_2 \in \mathbb{R})$

is the general solution of $(*)$.

Therefore $\{y_1, y_2\}$, where $y_1(x) = x$, $y_2(x) = 1$, for $x \in \mathbb{R}$,

is a fundamental set of solutions.

Example 2

Consider the ODE

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0,$$

on the interval $\mathrm{I} = (0, \infty)$.
Check that

$$y_1(x) = x, \qquad y_2(x) = x \log x,$$

form a **fundamental set of solutions**.

Step 1: check that y_1 and y_2 are solutions:

$$y_1'' - \frac{1}{x}y_1' + \frac{1}{x^2}y_1 = 0 - \frac{1}{x}\cdot 1 + \frac{1}{x^2}\cdot x = 0$$

Note $y_2' = \log x + 1$, $y_2'' = \frac{1}{x}$, so

$$y_2'' - \frac{1}{x}y_2' + \frac{1}{x^2}y_2 = \frac{1}{x} - \frac{1}{x}(\log x + 1) + \frac{1}{x^2}\cdot x \log x = 0.$$

So y_1 and y_2 are solutions.

Example 2 continued

Step 2: Check that y_1 and y_2 are linearly independent.

Suppose $c_1, c_2 \in \mathbb{R}$, such that

$$c_1 \cdot y_1(x) + c_2 y_2(x) = 0 \qquad (x \in (0,\infty))$$

$$\Rightarrow \quad c_1 \cdot x + c_2 \cdot x \log x = 0. \quad (1)$$

Evaluate (1) at $x = 1$:

$$c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0.$$

Evaluate (1) at $x = e$:

$$c_1 \cdot e + c_2 \cdot e = 0 \Rightarrow c_2 = 0.$$

So $c_1 = c_2 = 0$, and therefore $\{y_1, y_2\}$ are linearly indepedent and form a fundamental set of solutions.

Wronskians

To work out whether two solutions y_1 and y_2 of

$$y'' + p(x)y' + q(x)y = 0,$$

are **linearly independent**, we can make use of their **Wronskian** $\mathcal{W}(y_1, y_2)$, defined by

$$\mathcal{W}(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

This function is either

- **identically zero** on $\mathrm{I} \implies$ linear dependence,
- **nowhere zero** on $\mathrm{I} \implies$ linear independence.

Józef Maria Hoene-Wroński

Józef Maria Hoene-Wroński, by Laurent-Charles Maréchal

Born	Josef Hoëné 23 August 1776 Wolsztyn, Poznań Province, Poland
Died	9 August 1853 (aged 76) Neuilly-sur-Seine, France
Nationality	Polish

Abel's formula

The **Wronskian** $\mathcal{W}(y_1, y_2)(x)$ satisfies the first order ODE

$$W'(x) + p(x)W(x) = 0.$$

The general solution of this ODE is

$$W(x) = c\, e^{-\int p(x)dx},$$

where c an arbitrary constant.

Abel's formula

Take a point $x_0 \in \mathrm{I}$, then

$$\mathcal{W}(y_1, y_2)(x) = \mathcal{W}(y_1, y_2)(x_0)\, e^{-\int_{x_0}^{x} p(t)dt}$$

Niels Henrik Abel

Born	5 August 1802 Nedstrand, Denmark–Norway
Died	6 April 1829 (aged 26) Froland, Norway
Nationality	Norwegian

It follows from Abel's formula that $\mathcal{W}(y_1, y_2)(x)$ is either

- **identically zero** on $\mathrm{I} \implies$ linear dependence of $\{y_1, y_2\}$,
- **nowhere zero** on $\mathrm{I} \implies$ linear independence of $\{y_1, y_2\}$.

Linear independence

Theorem

Let $p(x)$ and $q(x)$ be continuous functions on an open interval $\mathrm{I} \subseteq \mathbb{R}$. Consider the ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{3}$$

For any solutions y_1 and y_2 of the ODE, the following are equivalent:

(1) $\{y_1, y_2\}$ is a **fundamental set of solutions**,

(2) y_1 and y_2 are **linearly independent**,

(3) $y = c_1\, y_1 + c_2\, y_2$, where $c_1, c_2 \in \mathbb{R}$, is the **general solution** of (3),

(4) the **Wronskian** $\mathcal{W}(y_1, y_2)$ is nonzero at every point in I,

(5) the **Wronskian** $\mathcal{W}(y_1, y_2)$ is nonzero at a point in I.

Example 2 revisited

Consider the ODE
$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0,$$
on the interval $\mathrm{I} = (0, \infty)$.
Compute the Wronskian of the solutions
$$y_1(x) = x, \qquad y_2(x) = x \log x,$$
and check that it satisfies $W' + p(x)W = 0$, where $p(x) = -\frac{1}{x}$.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x\log x \\ 1 & \log x + 1 \end{vmatrix} = x(\log x + 1) - x\log x = x$$

Note that the Wronskian is nonzero on $(0, \infty)$ and thus $\{y_1, y_2\}$ form a fundamental set of solutions.

Substitution of $W(y_1, y_2) = x$ into the ODE $W' + p(x)W = 0$, gives

$$W(y_1, y_2)' + p(x) \cdot W(y_1, y_2) = 1 + \left(-\frac{1}{x}\right) \cdot x = 0 \quad \checkmark$$

Example 2 revisited, continued

summary

Today's lecture was about 2nd order homogeneous linear ODEs.

After today's lecture, you should

- be familiar with **differential operators** and the **superposition principle**,
- understand what a **fundamental set of solutions** is,
- understand how to check **linear independence** of solutions through the computation of their **Wronskian**,
- know **Abel's formula**.