MATH2021 - Differential Equations
Week 8, Lecture 3

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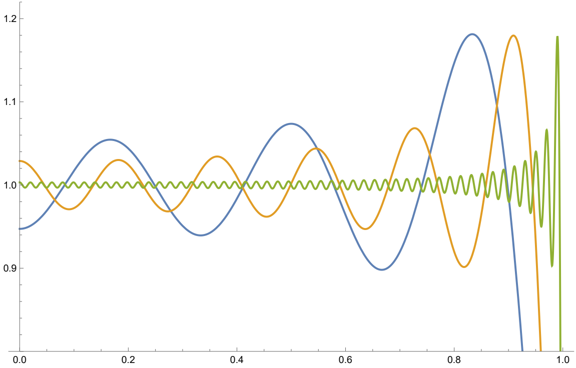

Past and present

- Last lecture:
  - separable ODEs
  - existence and uniqueness for first order ODEs
  - exact ODEs
- Today:
  - linear first order ODEs
  - variation of parameters method

Linear ODEs revisited

An ODE is called **linear** when it is linear in the unknown function $y(x)$ and its derivatives $y'(x), y''(x), y'''(x) \dots$.
Otherwise, the ODE is called **nonlinear**.

In other words, we can write a linear ODE of order n as

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = q(x),$$

for some coefficients $a_k(x)$, $0 \le k \le n-1$, and function $q(x)$.

If $q(x) \equiv 0$, the linear ODE is called **homogeneous**. Otherwise, it is called **inhomogeneous**.

Examples:

$y'' - \cos x \, y + 5 = 0$ $\left\{ \begin{array}{l} \text{2nd order,} \\ \text{linear,} \\ \text{inhomogeneous,} \end{array} \right.$ ODE

$y^{(5)} + x^3 y'' + e^x y = 0$ $\left\{ \begin{array}{l} \text{5th order,} \\ \text{linear,} \\ \text{homogeneous.} \end{array} \right.$

- A linear first order ODE can be written as
$$y' + p(x)y = q(x).$$
- If $q(x) \equiv 0$ it is **homogeneous**. Otherwise, it is **inhomogeneous**.
- We will assume that both $p(x)$ and $q(x)$ are continuous on an interval $\mathrm{I} \subseteq \mathbb{R}$.
- A solution of the ODE is a differentiable function $y : \mathrm{I} \to \mathbb{R}$ that satisfies
$$y'(x) + p(x)y(x) = q(x).$$
- Example:
$$y' + \frac{1}{x}y = \frac{1}{x^2}$$

We would take $I = (0, \infty)$.

A solution to the ODE is given by
$$y : I \to \mathbb{R},\ x \mapsto \frac{\log x}{x}.$$

Check: $y' + \frac{1}{x}y = \frac{1}{x^2} - \frac{\log x}{x^2} + \frac{1}{x} \cdot \frac{\log x}{x} = \frac{1}{x^2}.$ ✓

The homogeneous case

- Consider the homogeneous linear ODE

$$y' + p(x)y = 0. \quad (*) \qquad \frac{dy}{dx} = -p(x)\cdot y$$

- This ODE is **separable**. Indeed we can rewrite it as

$$\boxed{\frac{1}{y}\frac{dy}{dx} = -p(x)}$$

- Integrating both sides with respect to x gives

$$\log(y) = -\int p(x)\,dx + c, \qquad (c \text{ an arbitrary constant})$$

- Solving for y gives

$$y(x) = e^c e^{-\int p(x)\,dx}.$$

- Setting $\underline{C := e^c}$, we find the general solution

$$y(x) = C\,e^{-\int p(x)\,dx},$$

where $C \in \mathbb{R}$ arbitrary.

If we put $C = 0$, then $y(x) \equiv 0$ is a solution to $(*)$.

The set of solutions

The set of solutions to the homogeneous linear ODE

$$y' + p(x)y = 0,$$

is given by

$$V_{\text{hom}} = \{c\, e^{-\int p(x)\, dx} : c \in \mathbb{R}\}.$$

- Note that V_{hom} is a **vector space** over $\mathbb{R}$.
- The dimension of this vector space is **one**.
- More generally, the set of solutions to a homogeneous linear ODE of order n is an n-dimensional vector space over $\mathbb{R}$.

$$c_1 e^{-\int p(x)dx} + c_2 e^{-\int p(x)dx} = (c_1+c_2) e^{-\int p(x)dx}$$

The method of variation of parameters

To solve the general linear first order ODE,

$$y' + p(x)y = q(x),$$

we apply an **ansatz** that goes back to Lagrange and is called the method of **variation of parameters**.

ansatz ≈ educated guess

Joseph-Louis Lagrange

Born	Giuseppe Lodovico Lagrangia 25 January 1736 Turin, Kingdom of Sardinia
Died	10 April 1813 (aged 77) Paris, First French Empire

The ansatz

Recall that the solution of the homogeneous ODE when $q(x) \equiv 0$ is given by

$$y(x) = c\, y_{\text{hom}}(x), \qquad y_{\text{hom}}(x) = e^{-\int p(x)\,dx}.$$

where c is an arbitrary constant.

For the general ODE,

$$y' + p(x)y = q(x),$$

we try the **ansatz**

$$y(x) = c(x)\, y_{\text{hom}}(x)$$

where $c(x)$ depends on x!

Substitution

Recall $y_{\mathrm{hom}}'(x) = -p(x) y_{\mathrm{hom}}(x)$. ⊛

Substituting $y(x) = c(x)\, y_{\mathrm{hom}}(x)$ into the left-hand side of the ODE gives

$$\text{LHS} = y' + p(x)y = \big(c(x)\, y_{\mathrm{hom}}(x)\big)' + p(x) \cdot c(x)\, y_{\mathrm{hom}}(x)$$

$$= c'(x)\, y_{\mathrm{hom}}(x) + c(x)\, y_{\mathrm{hom}}'(x) + p(x)\, c(x)\, y_{\mathrm{hom}}(x)$$

we apply ⊛

$$= c'(x)\, y_{\mathrm{hom}}(x) + \underline{c(x)\big(-p(x)\, y_{\mathrm{hom}}(x)\big)} + \underline{p(x)\, c(x)\, y_{\mathrm{hom}}(x)}$$

$$= c'(x)\, y_{\mathrm{hom}}(x)$$

$\text{RHS} = q(x)$.

We require $c'(x)\, y_{\mathrm{hom}}(x) = q(x)$.

We thus require

$$c'(x) = \frac{q(x)}{y_{\mathrm{hom}}(x)}.$$

We therefore set

$$c(x) = \int \frac{q(x)}{y_{\mathrm{hom}}(x)} dx + d, \qquad (d \text{ an arbitrary constant})$$

The general solution

The **general solution** to

$c(x) = \int \frac{q(x)}{y_{hom}(x)} dx + d$

$$y' + p(x)y = q(x),$$

is given by

$$\begin{aligned} y(x) &= y_{\text{hom}}(x)\left(\int \frac{q(x)}{y_{\text{hom}}(x)}dx + d\right) \\ &= e^{-\int p(x)\,dx}\left(\int q(x)e^{\int p(x)\,dx}dx + d\right), \end{aligned}$$

where d an arbitrary constant.

Note that the structure of the general solution is

$$y(x) = y_p(x) + d\, y_{\text{hom}}(x),$$

where $y_p(x)$ is the **particular solution**

$$\boxed{y_p(x) = y_{\text{hom}}(x) \int \frac{q(x)}{y_{\text{hom}}(x)} dx.}$$

Example 1

Find the general solution to the ODE

$$y' - \frac{1}{x}y = \text{[redacted]} \cdot x \cdot \cos x$$

We consider the homogeneous ODE $y' - \frac{1}{x} y = 0$.

This ODE is separable: $\frac{1}{y} y' = \frac{1}{x}$.

Integrate both sides w.r.t. x: $\log y = \log x + c$

Solve for y: $y(x) = C \cdot x$.

Example 1 continued

Our ansatz for the inhomogeneous ODE is

$$y(x) = c(x) \cdot x$$

Substitute into LHS:

$$y' - \frac{1}{x} y = c'(x) \cdot x + \underline{c(x)\, 1} - \underline{\frac{1}{x} \cdot c(x) \cdot x}$$

$$= c'(x)\, x$$

Equate to RHS: $\quad c'(x) \cdot \cancel{x} = \cancel{x} \cdot \cos x .$

Integrate: $\quad c(x) = \sin x + d \quad$ (d arbitrary constant).

Putting it all together:

$$y(x) = x \cdot (\sin x + d)$$

is the general solution.

- Suppose an object with temperature $T(t)$ is in a room with an ambient temperature T_{amb} that we assume to be constant.
- Then **Newton's law of cooling** says that the rate of change $T'(t)$ is proportional to the temperature difference $T(t) - T_{\text{amb}}$.
- In other words, $T(t)$ satisfies the ODE

$$T'(t) = -k(T(t) - T_{\text{amb}}), \quad (*)$$

  for some $k > 0$.
- Given the initial temperature $T(0) = T_0$, what is the temperature at time t?

We write $(*)$ as $T'(t) + k \cdot T(t) = k \cdot T_{\text{amb}}$

We consider homogeneous ODE: $T'(t) + k \cdot T(t) = 0$.

The general solution is $T(t) = c \cdot e^{-kt}$,

c an arbitrary constant.

Newton's law of cooling, continued

Ansatz: $T(t) = c(t) \cdot e^{-kt}$

Substitute into

$$LHS = T' + k \cdot T = c'(t)\, e^{-kt}$$

$$RHS = k \cdot T_{amb}$$

Therefore

$$c(t) = \int \frac{k \cdot T_{amb}}{e^{-kt}}\, dt + d = \int k\, T_{amb}\, e^{kt}\, dt + d$$

$$= T_{amb} \cdot e^{kt} + d.$$

The general solution is $T(t) = e^{-kt} \cdot (T_{amb}\, e^{kt} + d)$

$$= T_{amb} + d\, e^{-kt}$$

Putting $t=0$, $T_0 = T_{amb} + d$,

so $T(t) = T_{amb} + (T_0 - T_{amb})\, e^{-kt}$.

A differential operator

Consider the **differential operator**

$$\mathcal{L}[y] = y' + p(x)y.$$

It takes in a differentiable function (on I) and outputs a function.

This operator is **linear**, that is,

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2], \quad \mathcal{L}[c\,y] = c\,\mathcal{L}[y],$$

for any $c \in \mathbb{R}$ and differentiable functions y_1, y_2 (on I).

Check:

Structure of general solution

The ODE

$$y' + p(x)y = q(x),$$

can be written as

$$\mathcal{L}[y] = q.$$

Given any particular solution y_p and any non-trivial solution y_{hom} to the homogeneous ODE

$$\mathcal{L}[y] = 0,$$

then the general solution is given by

$$y = y_p + d\, y_{\text{hom}} \qquad (d \text{ an arbitrary constant}).$$

Example 3: practice class exercise 7(b)

Find the solution to the initial value problem

$$\frac{y'}{x} + 3xy = x, \qquad y'(1) = 0.$$

Example 3 continued

summary

After today's lecture, you should be able to

- identify **homegeneous** and **inhomogeneous** linear ODEs,
- solve linear ODEs using the method of **variation of parameters**.

This is the end of week 8. Next week, we will study second order ODEs.