MATH2021 - Differential Equations
Week 8, Lecture 2

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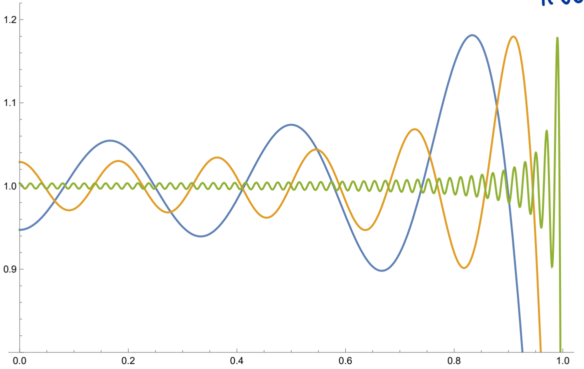


Past and present

- Last lecture:
    - general introduction to ordinary differential equations (ODEs)
    - order of an ODE
    - linear vs nonlinear ODEs
    - general and particular solutions
- Today:
    - separable ODEs
    - existence and uniqueness for first order ODEs
    - exact ODEs

Recap: precise definition of solutions

Consider a first order ODE in standard form,

$$y' = f(x, y),$$

where f is some function $f : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^2$ an open set.

- Let $I \subseteq \mathbb{R}$ be an interval.
- A function $y : I \to R$ is called a **solution** of the ODE if
  - (i) y is differentiable in I,
  - (ii) $(x, y(x)) \in D$ for $x \in I$, and
  - (iii) $y'(x) = f(x, y(x))$ for all $x \in I$.

For $(x_0, y_0) \in D$, the problem of finding a solution to

$$\boxed{y' = f(x, y),} \quad \text{with initial condition} \quad y(x_0) = y_0,$$

is known as an **initial value problem**.

Five types of 1st order ODEs

The five types of 1st order ODEs in standard form are

(1) $y' = f(x)$, f only depends on x, e.g. $y' = \cos x$

(2) $y' = f(y)$, f only depends on y, e.g. $y' = y^2$

(3) $y' = g(x)h(y)$, a **separable** ODE, e.g. $y' = (1 + \sin x)e^y$

→ **Note**: types (1) and (2) are special cases of type (3).

(4) $y' + p(x)y = q(x)$, **linear** ODE, e.g. $y' = 2xy + \cos(e^x)$

$y' = -p(x)y + q(x)$

(5) $y' = f(x, y)$, **nonlinear** ODE, e.g. $y' = \cos(x \cdot y) + e^y$

Type 1

- ODE: $y' = f(x)$
- By the **fundamental theorem of calculus**, this ODE always has a solution provided that $f(x)$ is continuous,

$$y(x) = \int f(x)dx + c,$$

where c an arbitrary constant.

- Example: $y' = x\,e^x$

The solution is given by

integration by parts

$$\begin{aligned} y(x) &= \int x e^x dx + c \\ &= x e^x - \int 1 \cdot e^x dx + c \\ &= x e^x - e^x + c \\ &= (x-1)e^x + c \end{aligned}$$

Type 2

- ODE: $y' = f(y)$ $\Leftrightarrow$ $\frac{dy}{dx} = f(y)$
- we rewrite the ODE as

$$\boxed{1 = \frac{1}{f(y)}\frac{dy}{dx}.}$$

- By the **fundamental theorem of calculus**, this ODE always has a solution provided that $\frac{1}{f(y)}$ is continuous,

$$x = \int \frac{1}{f(y)} dy + c,$$

where c an arbitrary constant.

- Here we get x as a function of y.
- Sometimes, it is possible to write y explicitly as a function of x.

$$\int \frac{1}{f(y)} \frac{dy}{dx} dx = \int \frac{1}{f(y)} dy$$

Type 2 example

- ODE: $y' = y^2$

divide by y^2:

integrate with respect to x

$$1 = \frac{1}{y^2}\frac{dy}{dx}$$

$$x = \int \frac{1}{y^2}\,dy + c$$

$$= -\frac{1}{y} + c$$

Solving for y gives

$$y(x) = \frac{1}{c - x},$$

where c an arbitrary constant.

Type 3, separable case

- ODE: $y' = g(x)h(y)$ $\Longleftrightarrow$ $\frac{dy}{dx} = g(x)\,h(y)$
- Such an ODE is called **separable**.
- we rewrite the ODE as
$$\boxed{\frac{1}{h(y)}\frac{dy}{dx} = g(x)}$$
- Integrating both sides, we obtain
$$\int \frac{1}{h(y)}dy = \int g(x)dx + c,$$
where c an arbitrary constant.
- In general, this gives an **implicit** expression for y as a function of x.
- Sometimes, it is possible to write y explicitly as a function of x.

Type 3 example

- ODE: $\cos(x)y' - y^2\tan(x) = 0$

Putting ODE in standard form

$$y' = y^2\,\frac{\tan x}{\cos x} = y^2\,\frac{\sin x}{\cos^2 x}$$

divide by y^2

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{\sin x}{\cos^2 x}$$

integrate both sides w.r.t. x

$$\int \frac{1}{y^2}\frac{dy}{dx}\,dx = \int \frac{1}{y^2}\,dy = -\frac{1}{y}$$

$$\|$$

$$\int \frac{\sin x}{\cos^2 x}\,dx = \frac{1}{\cos x} + c.$$

We obtain

$$-\frac{1}{y} = \frac{1}{\cos x} + c$$

So

$$y(x) = -\frac{1}{\frac{1}{\cos x} + c} = -\frac{\cos x}{1 + c\cos x}$$

Type 5: nonlinear ODEs

- ODE: $\quad y' = f(x,y)$,

  where f is **nonlinear** in y.

- There are methods for solving some special nonlinear ODEs.

- However, there is **no general method** to find explicit solutions to nonlinear ODEs.

- Nonetheless, there are theorems which guarantee **existence** and **uniqueness** of solutions.

- We now state such a theorem for **initial value problems** for 1st order ODEs.

Existence and uniqueness, part I

Theorem, part I: existence

Consider the ODE: $y' = f(x, y)$,
where f is some function $f : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^2$ an open set.

Let $(x_0, y_0) \in D$ and suppose there exists a rectangle

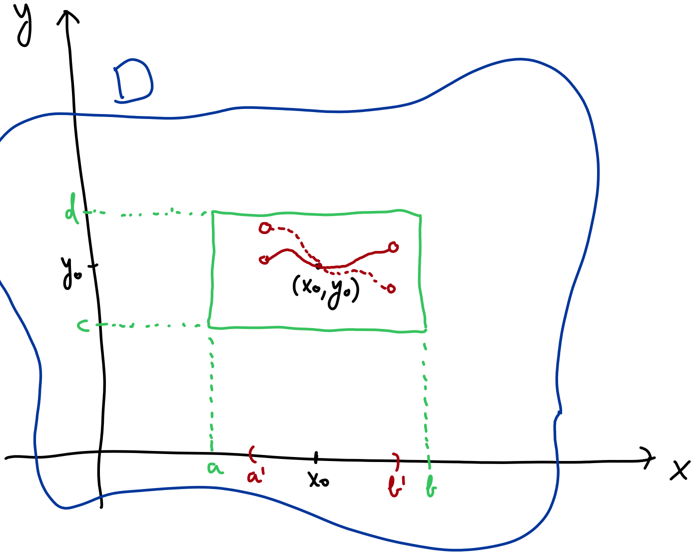
$$R = \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\} \subseteq D,$$

containing (x_0, y_0), such that f is continuous on R.
Then the **initial value problem**

$$\boxed{y' = f(x, y), \qquad y(x_0) = y_0,}$$

has **at least one solution** on an open subinterval $(a', b') \subseteq (a, b)$ containing x_0.

Illustration



Existence and uniqueness, part II

Theorem, part II: uniqueness

If f is differentiable with respect to y, and both f and f_y are continuous on the rectangle R, then the **initial value problem**

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

has **a unique solution** on an open subinterval $(a', b') \subseteq (a, b)$ containing x_0.

Example: $f(x, y) = \frac{1}{x} + \frac{1}{y}$.

$$f_y(x,y) = -\frac{1}{y^2}.$$

If $(x_0, y_0) \in \mathbb{R}^2$, $x_0, y_0 \neq 0$, then the initial value problem

$$y' = \frac{1}{x} + \frac{1}{y}, \qquad y(x_0) = y_0,$$

has a unique solution on some interval containing x_0.

Remarks on existence and uniqueness theorem

Part I is also known as the **Peano existence theorem**.

- It guarantees that a solution exists on some open interval containing x_0.
- However, it provides
  (a) no information on how to find a solution.
  (b) no information to determine the open interval on which it exists.
  (c) no information on the number of solutions.

Part II is on **uniqueness**.

- Under an additional condition, it guarantees that a unique solution exists on some open interval containing x_0.
- However, it does not provide any information to determine the open interval on which a unique solution exists.

Implicit solutions

- Consider the ODE

$$\frac{dy}{dx} = \frac{2x}{5y^5 - 2y + 1}. \iff (5y^5-2y+1)dy = 2x\,dx$$

- This is a separable equation, and we can rewrite it as

$$\boxed{2x\,dx - (6y^5 - 2y + 1)dy = 0.}$$

- Integrating both sides gives

$$x^2 - y^6 + y^2 - y = c \tag{1}$$

$$F(x,y) = x^2 - y^6 + y^2 - y$$

  for some constant c.
- It is difficult to solve this equation for $y = y(x)$.
- Very often, solutions to ODEs have to be left in the **implicit form**

$$F(x, y) = c, \qquad (c \text{ a constant}).$$

  We call this an **implicit solution** to the ODE.

Exact ODEs

- Suppose $y = y(x)$ satisfies
$$F(x,y) = c, \qquad (c \text{ a constant}),$$
where F is continuously differentiable.
- Differentiating with respect to x gives
$$F_x(x,y) + F_y(x,y)\frac{dy}{dx} = 0.$$
- Or equivalently $$\boxed{F_x(x,y)dx + F_y(x,y)dy = 0.}$$

Consider the ODE: $M(x,y)dx + N(x,y)dy = 0.$ $\Longleftrightarrow$ $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

We call this ODE **exact** if there exists a continuously differentiable function $F(x,y)$ on such that

$$M(x,y) = F_x(x,y), \qquad N(x,y) = F_y(x,y).$$

In such case, F is called the **potential function**.

Example of finding an exact ODE

Find an exact ODE that is implicitly solved by

$$e^{xy}\sin(x) = c \qquad (c \text{ a constant}).$$

$F(x,y) = e^{xy}\sin x$

The ODE is given by

$$F_x(x,y)\,dx + F_y(x,y)\,dy = 0$$

Computing the partial derivatives yields

$$(y e^{xy}\sin x + e^{xy}\cos x)\,dx + x e^{xy}\sin x\,dy = 0$$

The exactness condition

To determine whether or not an ODE is **exact**, we have the following useful theorem.

Theorem

Suppose, on some open set $D \subseteq \mathbb{R}^2$,

- $M(x,y)$ and $N(x,y)$ are continuous,
- $M(x,y)$ has a continuous partial derivative with respect to y,
- $N(x,y)$ has a continuous partial derivative with respect to x.

Then the ODE

$$\boxed{M(x,y)dx + N(x,y)dy = 0,}$$

is **exact** (on D) if and only if

$$M_y(x,y) = N_x(x,y), \tag{2}$$

for $(x,y) \in D$.

Equation (2) is called the **exactness condition**.

Example

Show that the ODE

$$y(e^x + y)dx + (e^x + 2xy + y)dy = 0$$

is **exact** and find an implicit solution.

$M(x,y) = ye^x + y^2$, $N(x,y) = e^x + 2xy + y$.

We compute partial derivatives:

$M_y = e^x + 2y$, $N_x = e^x + 2y$.

The exactness condition $M_y = N_x$ is satisfied, so the ODE is <u>exact</u>.

We look for a potential function $F(x,y)$,

$F_x = M = ye^x + y^2 \Rightarrow F = \int ye^x + y^2 dx + g(y) = ye^x + y^2 x + g(y)$

$F_y = N = e^x + 2xy + y \Rightarrow F = e^x \cdot y + xy^2 + \frac{1}{2}y^2 + h(x)$

So $F(x,y) = ye^x + y^2 x + \frac{1}{2}y^2$.

Therefore $ye^x + y^2 x + \frac{1}{2}y^2 = c$ (c an arbitrary constant) is an implicit solution to the ODE.

summary

After today's lecture, you should be able to

- Identify and solve **separable** ODEs,
- Understand the meaning and implications of the **existence** and **uniqueness** theorem for 1st order ODEs,
- Identify and solve **exact** ODEs.