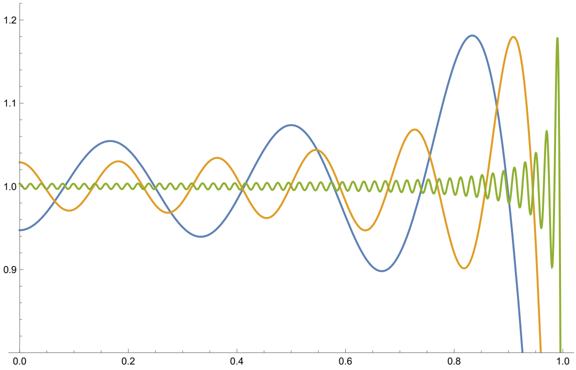
MATH2021 - Differential Equations
Week 8, Lecture 1

The University of Sydney

Lecturer : PIETER ROFFELSEN



Past and present

- Last week:
  - vector calculus

- Today:
  - general introduction to ordinary differential equations (ODEs)
  - First order ODEs

Differential equations

- Once you know how to do basic arithmetic with numbers, you can start studying **polynomial equations**, like

$$x^3 - 5x + 1 = 0, \qquad x^7 - 3x^4 + 2 = 0$$

- Being able to solve them can be very useful, though it is not always possible.
- Similarly, once you know how to differentiate functions and do arithmetic with them, you can start studying **differential equations**, like

$$y'(x)^2 - 5y(x) = x \qquad y'''(x) + 6y'(x)^3 - 5 = 0$$

- Here, the goal is to solve for $y(x)$ as a function of x.

An **ordinary differential equation** (ODE) is an equation involving an unknown function $y(x)$, as well as one or more of its derivatives $y'(x), y''(x), y'''(x), \ldots$

Examples:

- $y'(x) = y(x)$, $y(x) = e^x$ satisfies $y'(x) = y(x)$
- $y''(x) - 3xy(x) = \sqrt{x}$
- $y'(x)^2 - y(x)^3 + y(x) = 1.$

We often use short-hand notation like $y^{(3)} = y''' = y'''(x)$:

- $y'' - \sin(y) = 4x.$ $\longleftarrow$ $y''(x) - \sin(y(x)) = 4x$
- $y^{(7)} - 5x^3 y^{(4)} + e^x y = 0.$
- $y'' = x\,y$ (Airy equation)

Motivating example: throwing a ball upwards

Problem

- Suppose someone throws a ball upward at time $t = 0$ with initial speed $v_0 = 30\,m/s$.
- Let $h(t)$ denote the height of the ball in meters at time t, with $h(0) = 0$.
- How high will the ball get before coming back down?

Physics $\implies$ downward acceleration is given by $g \approx 9.81 m/s^2$.

In a formula: $h''(t) = -g$. $\leftarrow$ example of an ODE

integrate

$$h'(t) = \int -g\,dt = -gt + c_1$$

put $t=0$: $\quad v_0 = h'(0) = -g \cdot 0 + c_1 \Rightarrow c_1 = v_0$

integrate

$$h'(t) = -gt + v_0$$

$$h(t) = -\tfrac{1}{2}gt^2 + v_0 \cdot t + c_2 \xrightarrow{\text{Put } t=0} c_2 = 0$$

We know $h(t) = -\frac{1}{2}gt^2 + v_0 \cdot t$

To find maximum height: $0 = h'(t) = -gt + v_0 \Rightarrow t_{max} = \frac{v_0}{g}$

$$h_{max} = h(t_{max}) = \frac{1}{2}\frac{v_0^2}{g} \approx 46\,m$$

Throwing a ball upwards with air resistance

- Physics $\implies$ drag causes acceleration opposite to direction of travel proportional to speed squared.
- In a formula:

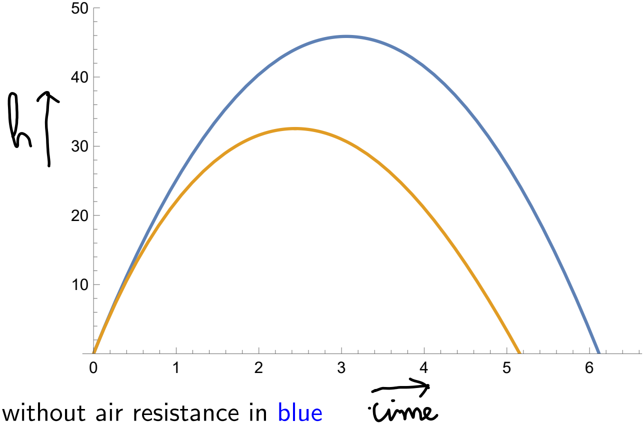
$$h''(t) = \begin{cases} -g - c\,h'(t)^2 & \text{if } h'(t) > 0, \\ -g & \text{if } h'(t) = 0, \\ -g + c\,h'(t)^2 & \text{if } h'(t) < 0, \end{cases}$$

  for some constant $c > 0$.
- More compactly,

$$h''(t) = -g - c\,|h'(t)| \cdot h'(t) \qquad \text{(ODE)}$$

Maximum height:	$h_{\max} = \frac{1}{2c} \log\left(1 + \frac{cv_0^2}{g}\right)$
At time:	$t_{\max} = \frac{1}{\sqrt{cg}} \tan^{-1}\left(\sqrt{\frac{c}{g}} v_0\right)$.

Comparison with $c = 1/100$



- without air resistance in blue
- with air resistance in orange

ODEs are useful to model various phenomena.

Solution of an ODE

Given an ODE, a function $y(x)$ is called a **solution** to the ODE when the equation of the ODE is identically satisfied when substituting $y(x)$ and its derivatives into the equation.

Example: check that $y(x) = e^{2x}$ is a solution to

$$y'' + y' - 6y = 0. \qquad \text{(ODE)}$$

$y = e^{2x}$

$y' = 2e^{2x}$

$y'' = 4e^{2x}$

Substitution into the ODE gives

$$\begin{aligned} y'' + y' - 6y &= 4e^{2x} + 2e^{2x} - 6e^{2x} \\ &= (4+2-6)e^{2x} \\ &= 0 \end{aligned}$$

So $y(x) = e^{2x}$ is a solution to the ODE.

Solution of an ODE

Example: check that $y(x) = x\cos(x)$ is a solution to

$$y''(x) + y(x) + 2\sin(x) = 0$$

$y = x\cos x$

$y' = \cos x - x\sin x$

$y'' = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x$

Substitute :

$y'' + y + 2\sin x = -2\sin x - x\cos x + x\cos x + 2\sin x = 0$

Order of an ODE

The **order** of an ODE is the order of the highest derivative appearing in the equation.

Examples:

$$y'' - (y')^2 + e^x y = 0 \qquad \text{second order ODE,}$$

$$y''' + 2\sin(x) y'' - y = 0 \qquad \text{third order ODE,}$$

$$(y')^2 + 4y^3 + 3y = 0 \qquad \text{first order ODE,}$$

$$y^{(n)} - x^5 y = 0 \qquad n\text{th order ODE,}$$

$$y^7 y' - 5y''' + 2 = 0$$

3 th order ODE.

General solution

Consider the simple ODE

$$y'' = 2x.$$

Integrating both sides gives

$$y' = x^2 + c_1$$

were c_1 is an arbitrary constant. Integrating again gives

$$y = \frac{1}{3}x^3 + c_1 x + c_2$$

This is the **general solution** of the **second** order ODE, with **two** arbitrary constants.

Given an ODE of order n, the **general solution** $y(x)$ of the ODE involves n arbitrary constants $c_1, c_2, \ldots, c_n$.

Examples:

order 1: $y' + y = 1$ general solution: $y(x) = 1 + c\, e^{-x}$

order 2: $y'' = y$ general solution: $y(x) = c_1 e^{x} + c_2 e^{-x}$

Particular solutions and initial conditions

Consider the ODE: $y'' = y$.

Find the **particular solution** satisfying the **initial conditions**

$$y(0) = 1, \quad y'(0) = 0.$$

General solution $y(x) = c_1 e^x + c_2 e^{-x}$ ⊛

Set $x=0$: $1 = y(0) = c_1 + c_2$

Differentiate ⊛ : $y'(x) = c_1 e^x - c_2 e^{-x}$

Set $x=0$: $0 = y'(0) = c_1 - c_2$

Hence $c_1 = c_2 = \frac{1}{2}$, so $y_p(x) = \frac{1}{2} e^x + \frac{1}{2} e^{-x} = \cosh(x)$

Given the general solution of an nth order ODE, a **particular solution** is obtained by assigning **particular** values to the arbitrary constants $c_1, c_2 \dots c_n$.

This is often done to meet conditions of the form

$$y(x_0) = a_0, \quad y'(x_0) = a_1, \quad y''(x_0) = a_2, \quad \dots \quad y^{(n-1)}(x_0) = a_{n-1},$$

called **initial conditions** at $x = x_0$.

Example

Find the particular solution of the ODE

$$y' + y = 1,$$

satisfying the initial condition $y(0) = 2$.

General solution is $y(x) = 1 + c e^{-x}$,
where c an arbitrary constant.

Setting $x = 0$ gives

$$2 = y(0) = 1 + c \cdot 1$$

So $c = 1$ and therefore

$$y(x) = 1 + e^{-x}$$

is the particular solution satisfying $y(0) = 2$

Linear vs non-linear

An ODE is called **linear** when it is linear in the unknown function $y(x)$ and its derivatives $y'(x), y''(x), y'''(x) \ldots$. Otherwise, the ODE is called **nonlinear**.

Examples:

$y'' - 2y' + e^x y = 0$	linear ODE
$y''' + \underline{yy'} = \cos(x)$	nonlinear ODE
$x^3 y'' + 2y = 0$	linear ODE
$(\underline{y^{(5)}})^2 + y + \cos x = 0$	nonlinear ODE
$y^{(5)} + x^2 y + \underline{\cos y} = 0$	nonlinear ODE

First order ODEs

- The **general** ODE of the first order has the **implicit form**

$$F(x, y, y') = 0,$$

for some function $F : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^3$ is an open set.

$$F(x, y, y') := y \cdot y' - \frac{1}{x} = 0 \qquad (*)$$

$$U = \{(x, y, y') \in \mathbb{R}^3 : x > 0\}$$

- If we can make y' the subject we obtain the **standard form**

$$y' = f(x, y),$$

where f is some function $f : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^2$ an open set.

Solve for y' in $(*)$: $\quad y' = \frac{1}{x \cdot y}$.

Here $f(x, y) = \frac{1}{xy}$ and $D = \{(x, y) \in \mathbb{R}^2 : x > 0, y \neq 0\}$.

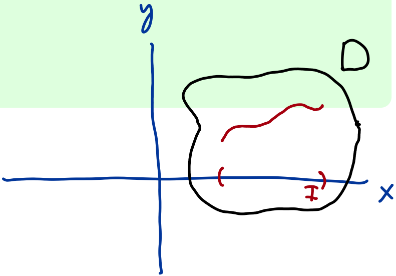
Precise definition of solutions

Consider a first order ODE in standard form,

$$y' = f(x, y),$$

where f is some function $f : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^2$ an open set.

- Let $I \subseteq \mathbb{R}$ be an interval.
- A function $y : I \to \mathbb{R}$ is called a **solution** of the ODE if
  - (i) y is differentiable in I,
  - (ii) $(x, y(x)) \in D$ for $x \in I$, and
  - (iii) $y'(x) = f(x, y(x))$ for all $x \in I$.



Five types of 1st order ODEs

The five types of 1st order ODEs in standard form are

(1) $y' = f(x)$, f only depends on x, e.g.

(2) $y' = f(y)$, f only depends on y, e.g.

(3) $y' = g(x)h(y)$, a **separable** ODE, e.g.

Note: types (1) and (2) are special cases of type (3).

(4) $y' + p(x)y = q(x)$, **linear** ODE, e.g.

(5) $y' = f(x, y)$, **nonlinear** ODE, e.g.

Type 1

- ODE: $y' = g(x)$
- By the **fundamental theorem of calculus**, this ODE always has a solution provided that $g(x)$ is continuous,

$$y(x) = \int g(x)dx + c,$$

where c an arbitrary constant.

- Example: $y' = x\,e^x$

Type 2

- ODE: $y' = h(y)$
- we rewrite the ODE as
$$1 = \frac{1}{h(y)} \frac{dy}{dx}.$$
- By the **fundamental theorem of calculus**, this ODE always has a solution provided that $\frac{1}{h(y)}$ is continuous,
$$x = \int \frac{1}{h(y)} dy + c,$$
where c an arbitrary constant.
- Here we get x as a function of y.

Type 2 example

- ODE: $y' = y^2$

Type 3

- ODE: $y' = g(x)h(y)$
- Such an ODE is called **separable**.
- we rewrite the ODE as
$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$
- Integrating both sides, we obtain
$$\int \frac{1}{h(y)}dy = \int g(x)dx + c,$$
where c an arbitrary constant.
- In general, this gives an **implicit** expression for y as a function of x.
- Sometimes, it is possible to write y explicitly as a function of x.

Type 3 example

- ODE: $\cos(x)y' - y^2 \tan(x) = 0$

summary

After today's lecture, you should be able to

- Explain what the **solution** to an ODE is.
- Identify the **order** of a differential equation.
- Write a first order ODE in **standard form**
- Solve **separable** ODEs.

Remainder of this week:

- In lecture 2, we study nonlinear first order ODEs (type 5)
- In lecture 3, we study linear first order ODEs (type 4)