MATH2021 - Differential Equations
Week 13, Lecture 3

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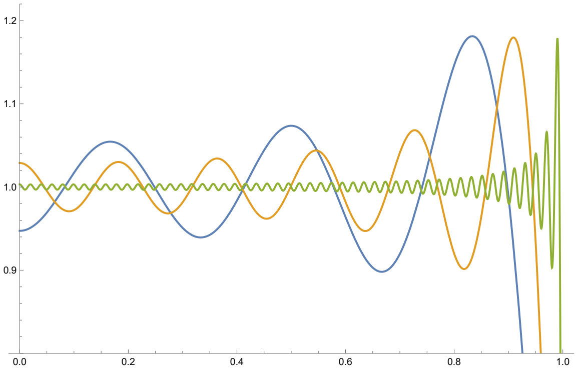


Past and present

- Last lecture:
  - The wave equation
  - modelling vibrations of a plucked string
  - travelling waves
- Today:
  - putting things in perspective
  - overview of all the DE techniques

Covered during the course

- 1st order ODEs
  - separable equations (often nonlinear)
  - exact equations (often nonlinear)
  - linear
  - nonlinear
- 2nd order ODEs
  - Linear
    - constant coefficients, homogeneous
    - constant coefficients, inhomogeneous
    - general coefficients
  - ~~nonlinear~~
- PDEs
  - heat equation
  - Laplace equation
  - wave equation

Not covered during the course

- ODEs of order > 2
- nonlinear ODEs of order > 1
- many PDEs:

Specific partial differential equations [edit]

- Broer–Kaup equations
- Burgers' equation
- Euler equations
- Fokker–Planck equation
- Hamilton–Jacobi equation, Hamilton–Jacobi–Bellman equation
- Heat equation
- Laplace's equation
  - Laplace operator
  - Harmonic function
  - Spherical harmonic
  - Poisson integral formula
- Klein–Gordon equation
- Korteweg–de Vries equation
  - Modified KdV–Burgers equation
- Maxwell's equations
- Navier–Stokes equations
- Poisson's equation
- Primitive equations (hydrodynamics)
- Schrödinger equation
- Wave equation

Expertise at USYD

Nalini Joshi is one the world's leading experts on **Painlevé equations**.

These are **nonlinear second order** ODEs whose solutions are **higher-order transcendental**.

The simplest example to write down is the **first Painlevé equation**,

$$y''(x) = 6\,y(x)^2 + x.$$

For the most difficult one, see next slide.

Nalini Joshi
AO

Nalini Joshi in 2014

Born	Yangon, Myanmar
Nationality	Australian
Alma mater	Princeton University
Known for	Research in integrable systems

The sixth Painlevé equation

The most difficult one to write down is the **sixth Painlevé equation**,

$$
\begin{aligned}
y'' = &\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)(y')^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y'+\\
&\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right).
\end{aligned}
$$

- Here $\alpha, \beta, \gamma, \delta$ are parameters.
- This equation was discovered in 1905 by Richard Fuchs.
- Applications in:
    - quantum field theory
    - general relativity
    - scattering of black holes

Navier–Stokes equations

Terence Tao has made major discoveries on the **Navier-Stokes equations**,

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla\rho+\nu\Delta\mathbf{v}+f(\mathbf{x},t).$$

These equations describe the flow of incompressible liquids like water.

It is a Clay Millennium Prize problem to show existence and smoothness of solutions with a US$1.000.000 reward.

Terence Tao

FAA FRS

Tao in 2021

Born	17 July 1975 (age 48) Adelaide, South Australia, Australia
Citizenship	Australia · United States[3]
Alma mater	Flinders University (BS, MSc) Princeton University (PhD)
Known for	Partial Differential Equations,

1st order ODEs

- **separable equations** (lecture 8-2)

$$y' = g(x)h(y)$$

- **exact equations** (lecture 8-2)

$$M(x,y)dx + N(x,y)dy = 0$$

- **linear** - Method of variation of parameters (lecture 8-3)

$$y' + p(x)\,y = f(x)$$

- **nonlinear** - existence and uniqueness (lecture 8-2)

$$y' = f(x,y)$$

1st order, separable (lecture 8-2)

- **separable** ODE: $y' = g(x)h(y)$

- we rewrite the ODE as
$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

- Integrating both sides, we obtain
$$\int \frac{1}{h(y)}dy = \int g(x)dx + c,$$

  where c an arbitrary constant.

- In general, this gives an **implicit** expression for y as a function of x.

- Sometimes, it is possible to write y explicitly as a function of x.

1st order, exact equations (lecture 8-2)

- The ODE
$$M(x,y)dx + N(x,y)dy = 0$$
is **exact** if there exists a continuously differentiable function $F(x,y)$ on such that
$$M(x,y) = F_x(x,y), \qquad N(x,y) = F_y(x,y).$$

- This holds true if and only if the **exactness condition**
$$M_y(x,y) = N_x(x,y),$$
is satisfied.

- The function $F(x,y)$ is called the **potential function**.

- The **implicit (general) solution** of the ODE is given by
$$F(x,y) = c,$$
where c an arbitrary constant.

- Linear ODE:

$$y' + p(x)\,y = f(x)$$

- If the **forcing function** $f(x) \equiv 0$, so

$$y' + p(x)y = 0,$$

  then the ODE is called **homogeneous** and the general solution is

$$y(x) = c\,e^{-\int p(x)\,dx}.$$

  where c an arbitrary constant.

- To solve the **inhomogeneous** ODE, we follow the **method of variation of parameters**, which suggests the **ansatz**

$$y_p(x) = c(x)y_{\text{hom}}(x), \qquad y_{\text{hom}}(x) := e^{-\int p(x)\,dx}.$$

- Substituting this ansatz gives

$$c'(x) = \frac{f(x)}{y_{\text{hom}}(x)}$$

  from which we can determine $c(x)$ and the general solution.

2nd order ODEs

- linear with constant coefficients
  - homogeneous:
    - **characteristic polynomial**
  - inhomogeneous:
    - **method of undetermined coefficients**
    - **method of variation of parameters**
    - **Laplace transform**
- linear with general coefficients
  - **method of variation of parameters**
  - **method of reduction of order**
  - **power series method**
- ~~nonlinear~~

2nd order, linear, constant coef., homogeneous (lec. 9-2)

- 2nd order linear homogeneous ODE with **constant coefficients**:

$$a\,y'' + b\,y' + c\,y = 0. \qquad a \neq 0$$

- The corresponding **characteristic polynomial** is given by

$$P(\lambda) = a\,\lambda^2 + b\,\lambda + c.$$

- Either:

(1) $P(\lambda)$ has two **distinct real** roots λ_1, λ_2, and the general solution of the ODE is

$$y(x) = c_1\,e^{\lambda_1 x} + c_2\,e^{\lambda_2 x}, \qquad (c_1, c_2 \in \mathbb{R}).$$

(2) $P(\lambda)$ has a **real double** root λ and the general solution of the ODE is

$$y(x) = e^{\lambda x}(c_1 + c_2\,x), \qquad (c_1, c_2 \in \mathbb{R}).$$

(3) $P(\lambda)$ has **distinct complex** roots

$$\lambda_1 = \alpha + \beta\,i, \qquad \lambda_2 = \alpha - \beta\,i,$$

and the general solution of the ODE is

$$y(x) = e^{\alpha\,x}\,(c_1 \cos \beta x + c_2 \sin \beta x)\,, \qquad (c_1, c_2 \in \mathbb{R}).$$

2nd order, linear, constant coef., inhomogeneous

- 2nd order linear inhomogeneous ODE with **constant coefficients**:

$$a\,y'' + b\,y' + c\,y = f(x).$$

- We have three methods available for such ODEs,
  - **method of undetermined coefficients** (lec. 9-3)
  - **method of variation of parameters** (lec. 10-1)
  - Method using the **Laplace transform** (lec. 10-3 & 11-1)
- General advise:
  - Use the method of undetermined coefficients if possible.
  - Else use the method of variation of parameters.
  - Only use the Laplace transform if you are asked to.

Table for method of undetermined coefficients (lec. 9-3)

$f(x)$	ansatz $y_p(x)$
c	A
$P_n(x)$	$A_0 + A_1 x + \ldots + A_n x^n$
$c\, e^{\alpha x}$	$A\, e^{\alpha x}$
$P_n(x)\, e^{\alpha x}$	$(A_0 + A_1 x + \ldots + A_n x^n) e^{\alpha x}$
$c_1 \cos\beta x + c_2 \sin\beta x$	$A \cos\beta x + B \sin\beta x$
$P_n(x)\,(c_1 \cos\beta x + c_2 \sin\beta x)$	$(A_0 + A_1 x + \ldots + A_n x^n)(A \cos\beta x + B \sin\beta x)$
$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$	$e^{\alpha x}(A \cos\beta x + B \sin\beta x)$
$P_n(x) e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$	$(A_0 + A_1 x + \ldots + A_n x^n) e^{\alpha x}(A \cos\beta x + B \sin\beta x)$

Here $P_n(x)$ is a polynomial of degree n in each case and $c, \alpha, \beta, c_1, c_2 \in \mathbb{R}$.

Modification rule: If any of the terms in the **ansatz** after expanding, is a solution of the homogeneous ODE $a\,y'' + b\,y' + c\,y = 0$, multiply the ansatz by x. If the same is still true, multiply by x again.

Method of undetermined coefficients (lecture 9-3)

To find a particular solution to the ODE

$$a\,y'' + b\,y' + c\,y = f(x), \tag{1}$$

the procedure of the method is as follows.

Step 1: Identify $f(x)$ as one of the forms in the left-hand side of the table. Take the corresponding ansatz on the right-hand side

$$y_p(x; \text{coefficients}),$$

appropriately modified if necessary. In the particular solution, there are one or more coefficients yet to be determined.

Step 2: Substitute the ansatz into the ODE (1) and determine the coefficients.

Method of variation of parameters (lecture 10-1)

The **method of variation of parameters** can be also be applied to linear second order ODEs with **non-constant coefficients**, so it is discussed a few slides further on.

Method using Laplace transform (lectures 10-3 & 11-1)

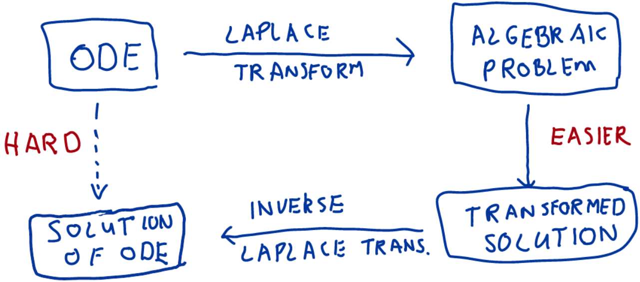
To solve an initial value problem of the form

$$a\,y'' + b\,y' + c\,y = f(x), \qquad y(0) = u, \quad y'(0) = v,$$

(1) apply **Laplace transform** to turn initial value problem into **algebraic problem** for $Y(s) = \mathcal{L}[y](s)$,

(2) solve algebraic problem for $Y(s)$,

(3) compute **inverse Laplace transform** of $Y(s)$, $y(x) = \mathcal{L}^{-1}[Y](x)$.



2nd order, linear, general coefficients

- 2nd order linear inhomogeneous ODE with ~~**constant**~~ general **coefficients**:

$$y'' + p(x)\,y' + q(x)\,y = f(x).$$

- We have three methods available for such ODEs,
  - **method of variation of parameters** (lec. 10-1)
    This method can only be used if you already have **two** linearly independent solutions of the homogeneous equation $y'' + p(x)\,y' + q(x)\,y = 0$.
  - **method of reduction of order** (lec. 10-1)
    This method can only be used if you already have **one** non-trivial solution of the homogeneous equation $y'' + p(x)\,y' + q(x)\,y = 0$.
  - **the power series method** (lec. 10-2)
    The only restriction to this method is that the coefficients of the ODE have to be nice enough (analytic).

Method of variation of parameters (lec. 10-1)

- Consider the ODE

$$y'' + p(x)\,y' + q(x)\,y = f(x).$$

- Suppose you know a **fundamental set of solutions** $\{y_1, y_2\}$ of the homogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

- Then the **method of variation of parameters** suggests the **ansatz**

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x), \quad \text{with} \quad u_1'\,y_1 + u_2'\,y_2 = 0.$$

- This defines a **particular solution** of the inhomogeneous ODE, provided that

$$u_1(x) = -\int \frac{y_2(x) f(x)}{\mathcal{W}(y_1, y_2)(x)}\,dx,$$
$$u_2(x) = \int \frac{y_1(x) f(x)}{\mathcal{W}(y_1, y_2)(x)}\,dx.$$

Method of reduction of order (lecture 10-1)

To find the general solution of

$$y'' + p(x)y' + q(x)y = f(x),$$

given that you know a solution $y_1(x)$ of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

follow the following steps.

(1) Substitute **ansatz** $y(x) = u(x)y_1(x)$ into the ODE to obtain a **1st order** ODE for $v := u'$.

(2) Find the general solution of the 1st order ODE for v.

(3) Integrate the general solution found in step (2) to find u.

(4) Substitute result for u into $y(x) = u(x)y_1(x)$ and simplify if possible. This is the **general solution**.

The power series method (lecture 10-2)

To apply the **power series method** around x_0 to a 2nd order linear ODE,

$$y'' + p(x)y' + q(x)y = f(x),$$

follow the following steps:

(0) If possible, rewrite the ODE in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x),$$

where $P_0(x), P_1(x), P_2(x), F(x)$ are polynomials. This step is not necessary nor always possible, but can greatly simplify the computations.

(1) Substitute a power series $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ into the ODE.

(2) Using the shifting of summation indices, ensure that all powers are aligned and read of the relations among the coefficients.

(3) recursively compute the coefficients.

Some remarks on the power series method

- The power series method can be applied to ODEs of **any order**, **linear** and **non-linear**!
- In particular, it can be applied to the **first Painlevé equation**,

$$y''(x) = 6\,y(x)^2 + x.$$

- It is part of a larger toolkit of so called **asymptotic methods** that are widely used in applied mathematics and physics.

PDEs

We looked at three PDEs in this course

- heat equation
- Laplace equation
- wave equation

The method of solution for the initial-boundary and boundary value problems that we encountered can be broken down into:

(1) Applying the method of **separation of variables**.

(2) Use some of the boundary conditions to obtain an **eigenvalue problem**.

(3) Solve the eigenvalue problem and corresponding ODE for the 'other variable'.

(4) Apply the **superposition principle** to construct a series solution with a countable number of arbitrary coefficients.

(5) Use the remaining initial/boundary condition(s) and **Fourier series** to determine the arbitrary coefficients.

End of the course