MATH2021 - Differential Equations
Week 13, Lecture 2

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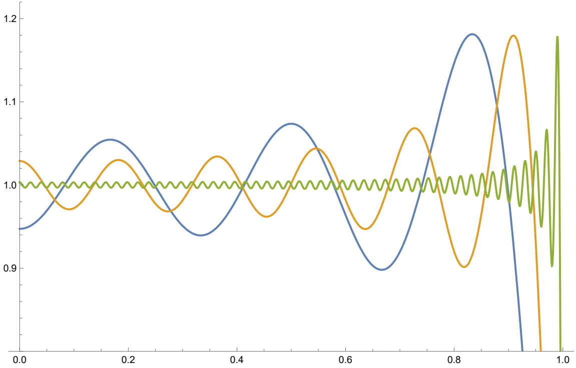


Past and present

- Last lecture:
  - Laplace's equation in polar coordinates
  - solving Laplace's equation on a disc
- Today:
  - The wave equation
  - modelling vibrations of a plucked string
  - travelling waves

The wave equation

- 1-D wave equation

$$u_{tt} = a^2\, u_{xx}$$

models *vibration of a stretched string* (link).

- 2-D wave equation

$$u_{tt} = a^2\,(u_{xx} + u_{yy})$$

models *water waves* and *vibration of a stretched membrane* (link).

- 3-D wave equation

$$u_{tt} = a^2\,(u_{xx} + u_{yy} + u_{zz})$$

models *electromagnetic waves* and *sound waves*.

The one dimensional wave equation

We will focus on the 1-D wave equation

$$u_{tt} = a^2\, u_{xx}\,,$$

where $a > 0$ is a parameter.

This equation was discovered by d'Alembert.

d'Alembert also derived that the **general solution** is given by

$$u(x,t) = F(x - a\,t) + G(x + a\,t),$$

where F and G are arbitrary twice differentiable functions.

Jean le Rond d'Alembert
FRS

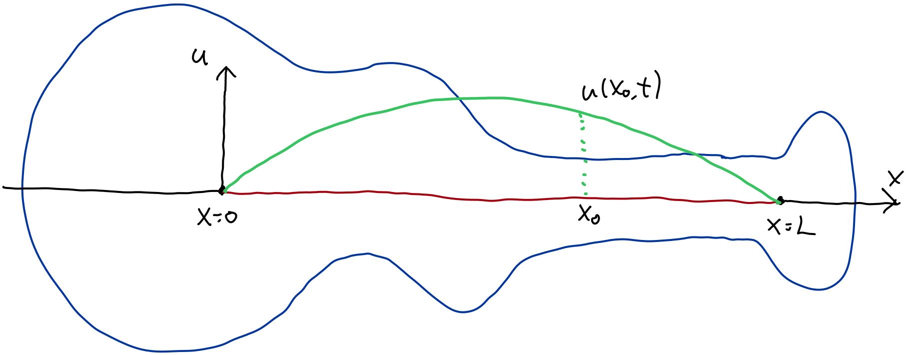
Pastel portrait of d'Alembert by Maurice Quentin de La Tour, 1753

Born	Jean-Baptiste le Rond d'Alembert 16 November 1717 Paris, France
Died	29 October 1783 (aged 65) Paris, France

A vibrating string

- Consider a string tightly stretched with **fixed endpoints** a distance $L > 0$ apart.
- We place the x-axis such that the two endpoints correspond to $x = 0$ and $x = L$.
- We assume that the string only vibrates in **one spatial direction** u, and denote by $u(x, t)$ the displacement from equilibrium at position x and time t.
- Assumptions from physics:
  - the mass density of the string is uniform.
  - the string is held under constant tension that is large enough such that other forces, such as gravity and air resistance, can be neglected.

Sketch



Boundary and initial conditions

- Under these conditions, time-evolution of the string is modelled by the one-dimensional **wave equation**

$$u_{tt} = a^2 \, u_{xx} ,$$

where $a > 0$ is a **parameter** that depends on the tension and mass density of the string.

- Fixed endpoints implies **boundary conditions**:

$$u(0, t) = 0, \qquad u(L, t) = 0 \qquad \text{for } t > 0.$$

- **Initial conditions**:

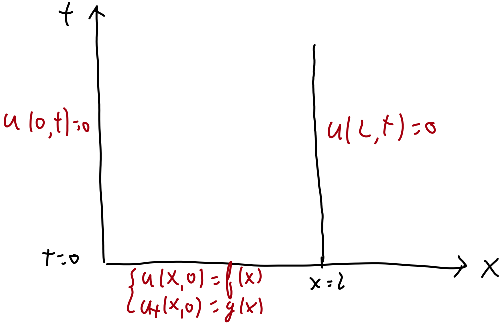
$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \qquad \text{for } 0 \le x \le L.$$

Initial-boundary value problem

Initial-boundary value problem

Solve the following **initial-boundary value problem** for the **wave equation**:

$$
\begin{cases}
u_{tt} = a^2\, u_{xx} & \text{for } (x,t) \in (0,L) \times (0,\infty), \\
u(0,t) = u(L,t) = 0 & \text{for } t \in (0,\infty), \\
u(x,0) = f(x) & \text{for } x \in [0,L], \\
u_t(x,0) = g(x) & \text{for } x \in [0,L].
\end{cases}
$$



The method of separation of variables

We look for solutions $u(x,t)$ that take the factorised form

$$u(x,t) = X(x)\,T(t).$$

Substitution into the wave equation $u_{tt} = a^2\,u_{xx}$ gives

$$X(x)\,T''(t) = a^2\,X''(x)\,T(t).$$

We can rewrite this as

$$\lambda := -\frac{X''(x)}{X(x)} = -\frac{T''(t)}{a^2\,T(t)},$$

where λ is independent of x, t and thus a **constant**.

So we obtain the differential equations

$$X''(x) + \lambda\,X(x) = 0, \qquad T''(t) + \lambda\,a^2\,T(t) = 0.$$

The boundary conditions

The boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad \text{for } t > 0,$$

imply

$$X(0)\,T(t) = 0, \quad X(L)\,T(t) = 0 \qquad \text{for } t > 0.$$

We do not want $T(t) \equiv 0$ and therefore we impose

$$X(0) = 0, \qquad X(L) = 0.$$

We have obtained the following **eigenvalue problem**

$$X''(x) + \lambda\,X(x) = 0, \qquad X(0) = 0, \quad X(L) = 0,$$

with the accompanying differential equation

$$T''(t) + \lambda\,a^2\,T(t) = 0.$$

Solutions to eigenvalue problem

From Lecture 11-2, we know that the solutions to this eigenvalue problem are given by

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2,$$

eigenfunction

where $n \geq 1$ integer.

The **general solution** of the corresponding differential equation

$$T''(t) + \lambda_n a^2 T(t) = 0, \qquad T''(t) + \left(\frac{n\pi a}{L}\right)^2 T(t) = 0$$

is given by

$$T_n(t) = A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right),$$

where A_n and B_n are arbitrary constants.

Superposition

For every integer $n \geq 1$,

$$u_n(x,t) = X_n(x)\,T_n(t) = \overbrace{\left(A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right)\right)}^{T_n(t)} \overbrace{\sin\left(\frac{n\pi x}{L}\right)}^{X_n(x)}$$

is a solution to the **wave equation**

$$u_{tt} = a^2\, u_{xx},$$

that satisfies the **boundary conditions**

$$u(0,t) = 0, \quad u(L,t) = 0, \qquad \text{for } t > 0.$$

Applying the superposition principle

Since the wave equation is **linear**, we can use **superposition** to construct the following series solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

which still satisfies the boundary conditions.

Initial conditions

The **first initial condition**, $u(x,0) = f(x)$ for $0 \le x \le L$, is equivalent to

$$\underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)}_{u(x,0)} = f(x) \qquad \text{for } 0 \le x \le L.$$

Since

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi a}{L}\left(\overbrace{-A_n \sin\left(\frac{n\pi a t}{L}\right)}^{=0 \text{ when } t=0} + \overbrace{B_n \cos\left(\frac{n\pi a t}{L}\right)}^{=B_n \text{ when } t=0}\right)\sin\left(\frac{n\pi x}{L}\right)$$

the **second initial condition**, $u_t(x,0) = g(x)$ for $0 \le x \le L$, is equivalent to

$$(*) \qquad \sum_{n=1}^{\infty} \left(\frac{n\pi a}{L} B_n\right) \sin\left(\frac{n\pi x}{L}\right) = g(x) \qquad \text{for } 0 \le x \le L.$$

Both of these are **Fourier sine series**.

$$\hat{B}_n = \frac{n\pi a}{L} B_n$$

$$(*) \quad \sum_{n=1}^{\infty} \hat{B}_n \sin\left(\frac{n\pi x}{L}\right) = g(x) \qquad \text{for } 0 \le x \le L$$

The complete solution

The infinite series

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

with coefficients

$$\begin{cases} A_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \\ B_n = \dfrac{L}{n\pi a} \hat{B}_n, \\ \hat{B}_n = \dfrac{2}{L} \displaystyle\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx, \end{cases}$$

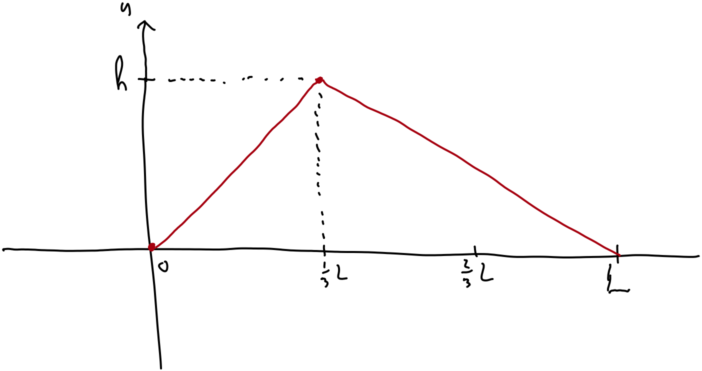
is the solution to the initial-boundary value problem.

Plucked string

We choose $g(x) = 0$ for $0 \leq x \leq L$ and set

$$f(x) = \begin{cases} \frac{3h}{L}x & \text{for } 0 \leq x \leq \frac{1}{3}L, \\ \frac{3h}{2L}(L - x) & \text{for } \frac{1}{3}L < x \leq L, \end{cases}$$

for an initial 'height' $h > 0$.



Computing coefficients

Since $g(x) = 0$ we know that $B_n = 0$ for all $n \geq 1$. We calculate the A_n's as follows

$$
\begin{aligned}
A_n =& \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
=& \frac{2}{L}\int_0^{\frac{1}{3}L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{\frac{1}{3}L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
=& \frac{2}{L}\int_0^{\frac{1}{3}L} \frac{3h}{L}x\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{\frac{1}{3}L}^{L} \frac{3h}{2L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\
=& \frac{6h}{L^2}\int_0^{\frac{1}{3}L} x\sin\left(\frac{n\pi x}{L}\right)dx \\
&+ \frac{3h}{L^2}\int_{\frac{1}{3}L}^{L} (L-x)\sin\left(\frac{n\pi x}{L}\right)dx
\end{aligned}
$$

Computing coefficients continued

$$= \frac{6h}{L^2}\left(\left[x\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{1}{3}L} - \int_0^{\frac{1}{3}L} 1\cdot\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

$$\frac{d}{dx}(L-x) = -1$$

$$+ \frac{3h}{L^2}\left(\left[(L-x)\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_{\frac{1}{3}L}^{L} - \int_{\frac{1}{3}L}^{L}(-1)\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{6h}{L^2}\left(\frac{1}{3}L\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi}{3}\right) - 0 + \left(\frac{L}{n\pi}\right)^2\left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{1}{3}L}\right)$$

$$+ \frac{3h}{L^2}\left(0 - \left(L - \tfrac{1}{3}L\right)\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi}{3}\right) - \left(\frac{L}{n\pi}\right)^2\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{\frac{1}{3}L}^{L}\right)$$

$$= -\frac{2h}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{6h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

$$+ \frac{2h}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{3h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) = \frac{9h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

The complete solution for the plucked string

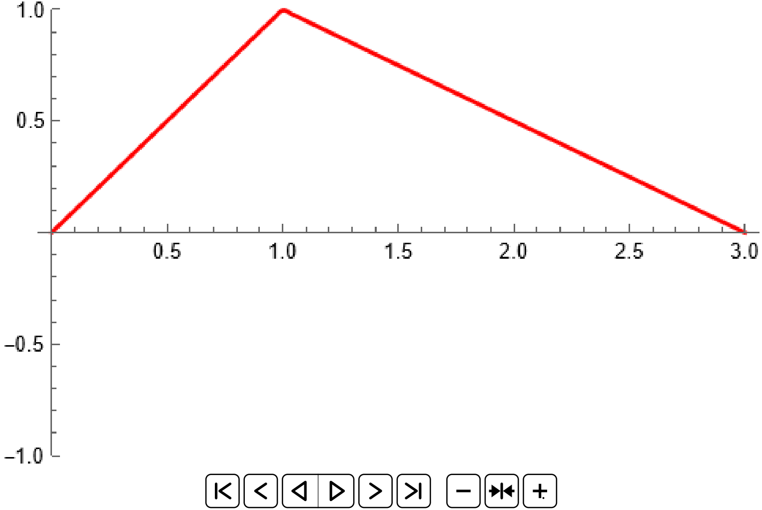
We found that the **coefficients** are given by

$$A_n = \frac{9\,h}{n^2\,\pi^2}\sin\left(\frac{n\,\pi}{3}\right), \qquad B_n = 0 \qquad \text{for } n \geq 1.$$

Therefore, the **complete solution** equals

$$u(x,t) = \sum_{n=1}^{\infty} \frac{9\,h}{n^2\,\pi^2}\sin\left(\frac{n\,\pi}{3}\right)\cos\left(\frac{n\,\pi\,a\,t}{L}\right)\sin\left(\frac{n\,\pi\,x}{L}\right)$$

Animation of solution with $h = 1$ and $L = 3$



Travelling waves

By the product to sum rule, $\cos A \sin B = \frac{1}{2}\left(\sin(B-A) + \sin(A+B)\right)$

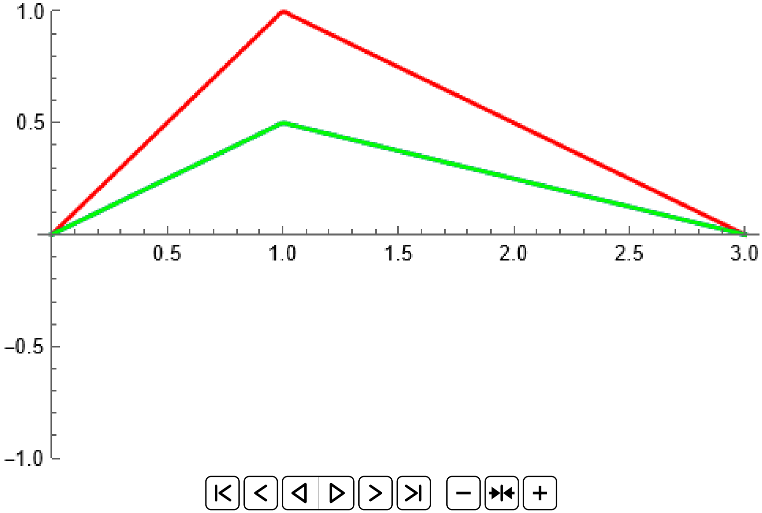
$$\cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\sin\left(\frac{n\pi}{L}(x-a t)\right) + \frac{1}{2}\sin\left(\frac{n\pi}{L}(x+a t)\right).$$

This means that the solution can be rewritten as

$$\begin{aligned}
u(x,t) =& \frac{1}{2}\sum_{n=1}^{\infty}\frac{9h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\left(\sin\left(\frac{n\pi}{L}(x-a t)\right) + \sin\left(\frac{n\pi}{L}(x+a t)\right)\right)\\
=& \overbrace{\frac{1}{2}\sum_{n=1}^{\infty}\frac{9h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\sin\left(\frac{n\pi}{L}(x-a t)\right)}^{F(x-at)}\\
&+ \overbrace{\frac{1}{2}\sum_{n=1}^{\infty}\frac{9h}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\sin\left(\frac{n\pi}{L}(x+a t)\right)}^{G(x+at)}\\
=& F(x-a t) + G(x+a t),
\end{aligned}$$

where $F(x-a t)$ and $G(x+a t)$ are so called **travelling waves**.

Animation of travelling waves with $h = 1$ and $L = 3$



- $u(x, t)$ in red
- $F(x - a\,t)$ in blue
- $G(x + a\,t)$ in green

summary

After today's lecture, you know

- how to solve the **one dimensional wave equation**.