# MATH2021 - Differential Equations
# Week 13, Lecture 1

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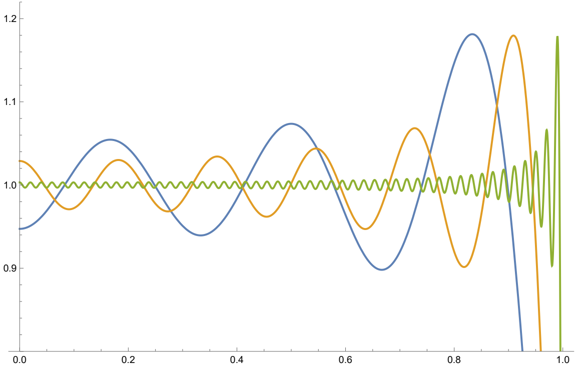


# Past and present

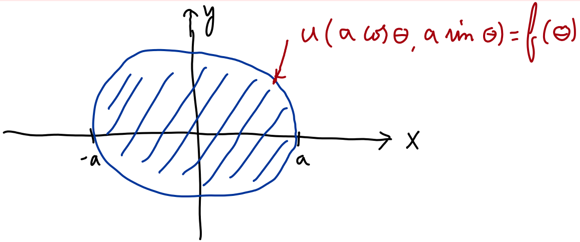
- Last lecture:
  - applications of Laplace's equation
  - solving Laplace's equation on a rectangle
- Today:
  - Laplace's equation in polar coordinates
  - solving Laplace's equation on a disc

Consider a disc of radius $a > 0$ and let $f(\theta)$ be a $2\pi$-periodic function.

**Boundary value problem on a disc**

Solve the following **boundary value problem** for **Laplace's equation** on a **disc**:

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{for } x^2 + y^2 < a^2, \\ u(a\cos\theta, a\sin\theta) = f(\theta), & \text{for } -\pi \le \theta \le \pi. \end{cases}$$




To solve this problem, we first transform the problem using **polar coordinates**.

# Laplace's equation in polar coordinates

- Recall **Laplace's equation**

$$u_{xx} + u_{yy} = 0.$$

- To study steady-state temperature distributions on a **disc**, we change to **polar coordinates**,

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \quad a \geq r > 0, \quad -\pi \leq \theta < \pi.$$

- We are going to show that

$$\Delta u = u_{xx} + u_{yy} = \frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta}.$$

- Therefore, in polar coordinates, Laplace's equation becomes

$$\frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta} = 0.$$

# Partial derivatives with respect to the radius

For the sake of clarity, we write

$$U(r,\theta) = u(x,y) = u(r\cos\theta, r\sin\theta), \qquad x = r\cos\theta, \quad y = r\sin\theta.$$

Let's compute some partial derivatives:

$$\begin{aligned} U_r &= u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r} \\ &= u_x \cos\theta + u_y \sin\theta. \end{aligned}$$

$$\begin{aligned} U_{rr} &= \frac{\partial}{\partial r}\left(u_x \cos\theta + u_y \sin\theta\right) \\ &= \frac{\partial}{\partial r}\left(u_x \cos\theta\right) + \frac{\partial}{\partial r}\left(u_y \sin\theta\right) \\ &= \left(u_{xx} \frac{\partial x}{\partial r} \cos\theta + u_{xy} \frac{\partial y}{\partial r} \cos\theta\right) + \left(u_{xy} \frac{\partial x}{\partial r} \sin\theta + u_{yy} \frac{\partial y}{\partial r} \sin\theta\right) \\ &= \left(u_{xx} \cos^2\theta + u_{xy} \sin\theta \cos\theta\right) + \left(u_{xy} \cos\theta \sin\theta + u_{yy} \sin^2\theta\right) \\ &= u_{xx} \cos^2\theta + u_{yy} \sin^2\theta + 2\, u_{xy} \cos\theta \sin\theta \end{aligned}$$

$u_x = u_x(x,y)$
$u_y = u_y(x,y)$

# Partial derivatives with respect to the angle

$$U_\theta = u_x \frac{\partial x}{\partial \theta} + u_y \frac{\partial y}{\partial \theta}$$

$$= -u_x\, r \sin\theta + u_y\, r \cos\theta.$$

$$\frac{\partial}{\partial \theta}(u_y\, r \cos\theta) =$$

$$\underbrace{\left(\frac{\partial}{\partial \theta} u_y\right) r \cos\theta}_{} + \underline{u_y\,(-r\sin\theta)}$$

$$= \left(u_{xy} \frac{\partial x}{\partial \theta} + u_{yy} \frac{dy}{d\theta}\right) \cdot r\cos\theta - u_y\, r \sin\theta$$

$$U_{\theta\theta} = \frac{\partial}{\partial \theta}\left(-u_x\, r \sin\theta + u_y\, r \cos\theta\right)$$

$$= \frac{\partial}{\partial \theta}\left(-u_x\, r \sin\theta\right) + \underline{\frac{\partial}{\partial \theta}\left(u_y\, r \cos\theta\right)}$$

$$= \left(u_{xx}\, r^2 \sin^2\theta - u_{xy}\, r^2 \sin\theta\cos\theta - u_x\, r\cos\theta\right)$$

$$+ \left(\underbrace{-u_{xy}\, r^2 \sin\theta\cos\theta + u_{yy}\, r^2 \cos^2\theta}_{} - \underline{u_y\, r \sin\theta}\right)$$

$$= u_{xx}\, r^2 \sin^2\theta + u_{yy}\, r^2 \cos^2\theta - 2\, u_{xy}\, r^2 \sin\theta\cos\theta - u_x\, r\cos\theta - u_y\, r \sin\theta.$$

# Putting it together

$$
\begin{aligned}
U_{rr} + \frac{1}{r^2} U_{\theta\theta} =& u_{xx} \cos^2\theta + u_{yy} \sin^2\theta + \cancel{2\, u_{xy} \cos\theta \sin\theta} \\
&+ u_{xx} \sin^2\theta + u_{yy} \cos^2\theta - \cancel{2\, u_{xy} \sin\theta \cos\theta} - u_x \frac{\cos\theta}{r} - u_y \frac{\sin\theta}{r} \\
=& u_{xx} + u_{yy} - u_x \frac{\cos\theta}{r} - u_y \frac{\sin\theta}{r}.
\end{aligned}
$$

Recall that $U_r = u_x \cos\theta + u_y \sin\theta$, therefore

$$
\begin{aligned}
\frac{1}{r} U_r + U_{rr} + \frac{1}{r^2} U_{\theta\theta} &= \frac{1}{r} \left( u_x \cos\theta + u_y \sin\theta \right) + u_{xx} + u_{yy} - u_x \frac{\cos\theta}{r} - u_y \frac{\sin\theta}{r} \\
&= u_{xx} + u_{yy}.
\end{aligned}
$$

# Steady-state temperature distribution on a disc

- Consider a **disc** $D$ of radius $a$:

$$D = \{(r\cos\theta, r\sin\theta) : 0 \le r \le a, -\pi \le \theta \le \pi\}.$$

- We look for a solution $u(r,\theta)$ of Laplace's equation

$$\frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta} = 0.$$

- We impose the **boundary condition**:

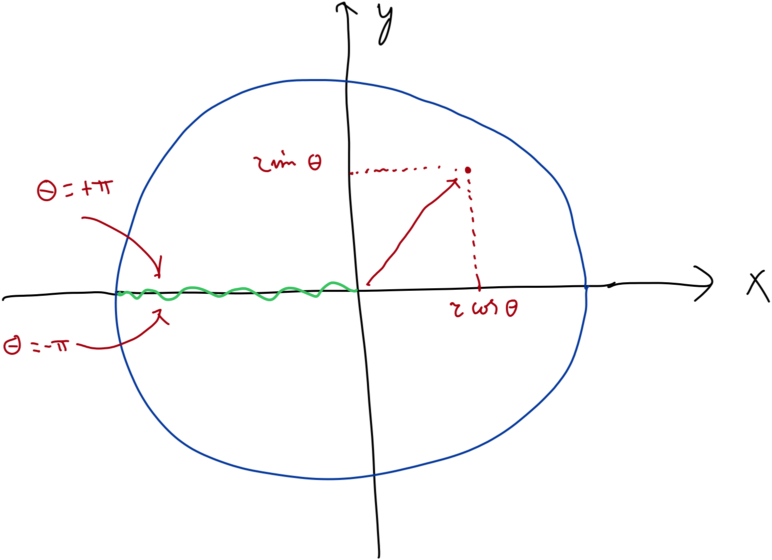
$$u(a,\theta) = f(\theta) \qquad -\pi \le \theta \le \pi.$$

- We further impose **periodic boundary conditions**

$$\begin{cases} u(r,-\pi) = u(r,\pi), \\ u_\theta(r,-\pi) = u_\theta(r,\pi), \end{cases} \qquad \text{for } 0 < r < a.$$

- **Hidden boundary condition**: we want $u(r,\theta)$ to be continuous at $r = 0$.

# Sketch



# Applying the method of separation of variables

- We first look for solutions that can be written as

$$u(r,\theta) = R(r)G(\theta).$$

- Note that

$$u_r = R'(r)G(\theta), \quad u_{rr} = R''(r)G(\theta), \quad u_{\theta\theta} = R(r)G''(\theta).$$

- So Laplace's equation becomes

$$\left(\frac{1}{r}R'(r) + R''(r)\right)G(\theta) + \frac{1}{r^2}R(r)G''(\theta) = 0.$$

divide by $\frac{1}{r^2} R(r) G(\theta)$

- We rewrite this equation as

$$-\frac{G''(\theta)}{G(\theta)} = \frac{r\,R'(r) + r^2\,R''(r)}{R(r)} =: \lambda.$$

- $\lambda$ does not depend on $r$ nor $\theta$ and is thus a **constant**.
- Consequently, we obtain the following ODEs

$$G''(\theta) + \lambda\,G(\theta) = 0, \qquad r^2\,R''(r) + r\,R'(r) - \lambda\,R(r) = 0$$

# Boundary conditions for $G$

$u(r,-\pi) = u(r,\pi) \qquad 0 < r < a$

- The **periodic boundary conditions** imply

$$R(r)G(-\pi) = R(r)G(\pi), \quad R(r)G'(-\pi) = R(r)G'(\pi) \qquad \text{for } r \in (0,a).$$

- We are not interested in $R(r) \equiv 0$, and thus we impose

$$G(-\pi) = G(\pi), \quad G'(-\pi) = G'(\pi).$$

- Combined with the ODE for $G(\theta)$, this yields the following **eigenvalue problem**

$$G''(\theta) + \lambda\, G(\theta) = 0, \qquad G(-\pi) = G(\pi), \quad G'(-\pi) = G'(\pi).$$

- We solved this eigenvalue problem in Lecture 12-2.

# Eigenvalue $\lambda = 0$

- $\lambda_0 = 0$ is an **eigenvalue** with **eigenfunction** $G_0(\theta) = A_0$.
- The corresponding ODE for $R(r)$ reads
$$r^2 R''(r) + r R'(r) = 0, \implies R''(r) + r^{-1} R'(r) = 0.$$
- This is an ODE of order 1 for $R'(r)$, with general solution
$$R'(r) = C e^{-\int r^{-1} dr} = C e^{-\log r} = \frac{C}{r}.$$
- Integrating, gives
$$R(r) = C \log(r) + D \qquad (C, D \text{ arbitrary constants}).$$
- We require $R(r)$ to be continuous at $r = 0$, so $C = 0$ and we may thus pick $R_0(r) = 1$.
- So
$$u_0(r, \theta) = G_0(\theta) R_0(r) = A_0$$
is a solution to Laplace's equation on the disc, satisfying the **periodic** and **hidden** boundary conditions.

# Positive eigenvalues

- For every $n \geq 1$, $\lambda_n = n^2$ is an **eigenvalue** with corresponding **eigenfunctions**

$$G_n(\theta) = A_n \cos(n\,\theta) + B_n \sin(n\,\theta),$$

  where $A_n$ and $B_n$ arbitrary constants.

- The corresponding ODE for $R(r)$,

$$r^2\,R''(r) + r\,R'(r) - n^2\,R(r) = 0,$$

  is an example of a **Cauchy-Euler equation**.

- Such equations can be solved through the **ansatz** $R(r) = r^\alpha$.

# Solving the Cauchy-Euler equation

Find the **general solution** of the ODE

$$r^2 R''(r) + r R'(r) - n^2 R(r) = 0.$$

*Hint: try the ansatz* $R(r) = r^\alpha$.

$R'(r) = \alpha r^{\alpha-1}$, $\quad R''(r) = \alpha(\alpha-1) r^{\alpha-2}$

Substitution gives:

$$\begin{aligned} 0 &= r^2 R'' + r R' - n^2 R \\ &= \alpha(\alpha-1) r^\alpha + \alpha r^\alpha - n^2 r^\alpha \\ &= (\alpha(\alpha-1) + \alpha - n^2) r^\alpha \\ &= (\alpha^2 - n^2) r^\alpha \end{aligned}$$

So $\alpha = n$ or $\alpha = -n$.

The general solution is:

$R(r) = C r^n + D r^{-n}$, $\quad$ ($C, D$ arbitrary constants)

# Determining $R_n$

- The **general solution** of the ODE for $R(r)$ is

$$R(r) = C\, r^n + D\, r^{-n}.$$

- Due to the **hidden** boundary condition, $D = 0$ and we thus pick $R_n(r) = r^n$.

- We conclude that

$$u_n(r,\theta) = G_n(\theta)R_n(r) = A_n\, r^n \cos(n\,\theta) + B_n\, r^n \sin(n\,\theta)$$

is a solution to Laplace's equation on the disc, satisfying the **periodic** and **hidden** boundary conditions.

# Superposition

**Applying the superposition principle**

Through **superposition**, we obtain the following solution given by an infinite series,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n\, r^n \cos(n\theta) + B_n\, r^n \sin(n\theta),$$

which satisfies the periodic and hidden boundary conditions.

The final boundary condition at the edge of the disc, $u(a,\theta) = f(\theta)$, for $-\pi \leq \theta \leq \pi$, gives

$$A_0 + \sum_{n=1}^{\infty} (A_n\, a^n) \cos(n\theta) + (B_n\, a^n) \sin(n\theta) = f(\theta) \qquad \text{for } -\pi \leq \theta \leq \pi.$$

We can thus compute the $A_n$'s and $B_n$'s using the standard formulas for **Fourier coefficients**.

# Complete solution to boundary value problem

The complete solution to the boundary value problem is given by

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n\, r^n \cos(n\,\theta) + B_n\, r^n \sin(n\,\theta),$$

with **coefficients**

$$A_0 = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta,$$

$$A_n = \frac{1}{\pi\, a^n} \int_{-\pi}^{\pi} f(\theta) \cos(n\,\theta)\, d\theta,$$

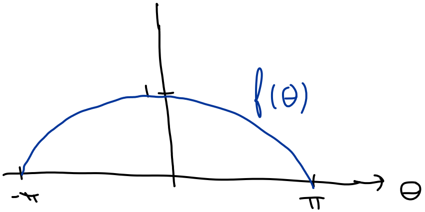
$$B_n = \frac{1}{\pi\, a^n} \int_{-\pi}^{\pi} f(\theta) \sin(n\,\theta)\, d\theta,$$

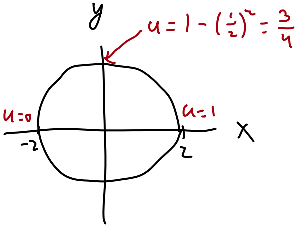
for $n \geq 1$.

# Example

We set the radius $a = 2$ and choose the temperature distribution at the edge of the disc to be

$$f(\theta) := 1 - \left(\frac{\theta}{\pi}\right)^2 \qquad \text{for } -\pi \leq \theta \leq \pi.$$




**Boundary value problem on a disc**

Solve the following **boundary value problem** for **Laplace's equation** on a disc:

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{for } x^2 + y^2 < 2^2, \\ u(2\cos\theta, 2\sin\theta) = 1 - \left(\frac{\theta}{\pi}\right)^2, & \text{for } -\pi \leq \theta \leq \pi. \end{cases}$$

# Computing coefficients

$$B_n = \frac{1}{\pi 2^n} \int_{-\pi}^{\pi} \underbrace{\left(1 - \left(\frac{\theta}{\pi}\right)^2\right) \sin(n\theta)}_{\text{odd function}} d\theta = 0$$

$$A_0 = \frac{2}{3}$$

$$A_n = \frac{4(-1)^{n+1}}{\pi^2 2^n n^2} \qquad n \geq 1$$

# Computing coefficients continued

# The solution

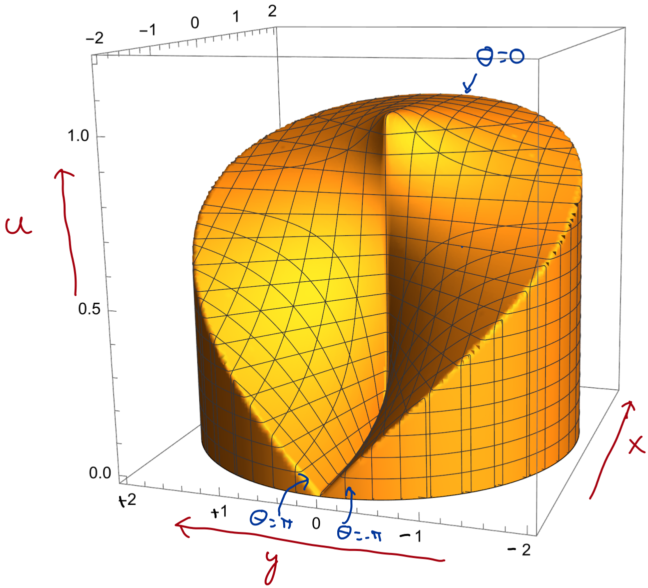
The solution to the example boundary value problem is given by

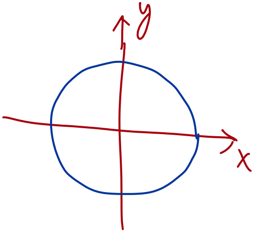
$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n\, r^n \cos(n\,\theta) + B_n\, r^n \sin(n\,\theta)$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi^2\, n^2\, 2^n}\, r^n \cos(n\,\theta)$$

$$= \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left(\frac{r}{2}\right)^n \cos(n\,\theta).$$

# Plot of solution



# summary

After today's lecture, you know

- Laplace's equation in **polar coordinates**.
- how to solve Laplace's equation on a disc.