MATH2021 - Differential Equations
Week 12, Lecture 3

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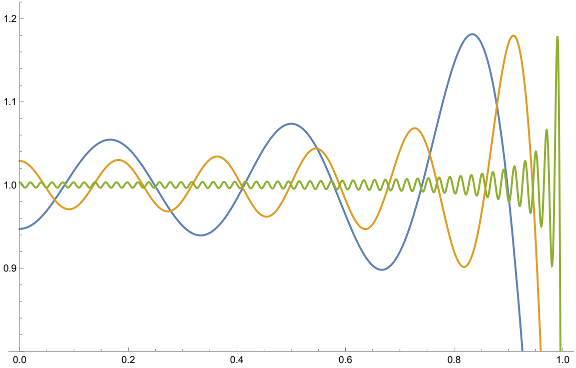


Past and present

- Last lecture we learnt how to solve several initial-boundary value problems for the heat equation.
- Today is all about Laplace's equation:
  - applications
  - solving the equation on a rectangle with boundary conditions.

Laplace's equation

In two dimensions, **Laplace's equation** is the PDE:

$u(x,y)$:

$$u_{xx} + u_{yy} = 0.$$

Similarly, in three dimensions:

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

And, in n dimensions:

$$u_{x_1 x_1} + \ldots + u_{x_n x_n} = 0.$$

It is often written as

$$\Delta u = 0,$$

where Δ is the **Laplace operator:**

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}.$$

Pierre-Simon Laplace

Pierre-Simon Laplace as chancellor of the Senate under the First French Empire

Born	23 March 1749 Beaumont-en-Auge, Normandy, Kingdom of France
Died	5 March 1827 (aged 77) Paris, Kingdom of France

Applications

Applications:

- steady state temperature distributions on metal surfaces.
- irrotational fluid flow in 2 and 3 dimensions.
- time independent electric fields.

We are going to use Laplace's equation to find steady state temperature distributions on metal plates of **rectangular** and **circular** shapes.

Steady-state distributions

Consider the **heat equation** with **two** spatial dimensions,

$$u_t = k(u_{xx} + u_{yy}).$$

Here $u(x, y, t)$ gives the temperature at position (x, y) and time t.

- As time evolves, the temperature distribution often goes to a **steady-state distribution**.
- A **steady state distribution** means a temperature distribution that does not change in time, that is, $u_t = 0$.
- The heat equation then reduces to Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

- For this reason, Laplace's equation can be used to model steady state temperature distributions of for instance metal surfaces.

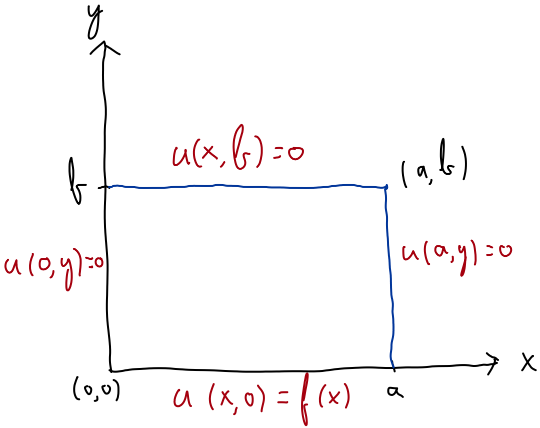
Temperature distributions on a rectangular plate

- Consider a flat rectangular plate of size $a \times b$.
- We use spatial coordinates (x, y), where

$$0 \leq x \leq a, \qquad 0 \leq y \leq b,$$

  on the rectangular plate, see sketch on next page.
- We further impose boundary conditions:
  - $u(0, y) = 0$ for $0 \leq y \leq b$ (left edge)
  - $u(a, y) = 0$ for $0 \leq y \leq b$ (right edge)
  - $u(x, b) = 0$ for $0 \leq x \leq a$ (top edge)
  - $u(x, 0) = f(x)$ for $0 \leq x \leq a$ (bottom edge)
- We want to know the temperature $u(x, y)$ for (x, y) inside the plate.

Sketch



Boundary value problem

Solve the following **boundary value problem** for **Laplace's equation** on a **rectangle**:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } (x, y) \in (0, a) \times (0, b), \\ u(0, y) = u(a, y) = 0 & \text{for } y \in (0, b), \\ u(x, b) = 0 & \text{for } x \in [0, a], \\ u(x, 0) = f(x) & \text{for } x \in [0, a]. \end{cases}$$

Applying the method of separation of variables

- We first look for solutions that can be written as

$$u(x,y) = X(x)Y(y).$$

- Note that

$$u_{xx} = X''(x)Y(y), \qquad u_{yy} = X(x)Y''(y).$$

- So Laplace's equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

- We rewrite this equation as

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} =: \lambda.$$

- λ does not depend on x nor y and is thus a **constant**.
- Consequently, we obtain the following ODEs

$$X''(x) + \lambda X(x) = 0, \qquad Y''(y) - \lambda Y(y) = 0.$$

Boundary conditions for X

- The **boundary conditions** $u(0, y) = u(a, y) = 0$ for $y \in (0, b)$, imply

$u(x,y) = X(x) \cdot Y(y)$

$$X(0)Y(y) = 0, \quad X(a)Y(y) = 0 \qquad \text{for } y \in (0, b).$$

- We are not interested in $Y(y) \equiv 0$, and thus we impose

$$X(0) = 0, \quad X(a) = 0.$$

- Combined with the ODE for $X(x)$, this yields the following **eigenvalue problem**

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \quad X(a) = 0.$$

- We worked out the solutions to this eigenvalue problem in Lecture 11-2. The **eigenvalues** and corresponding **eigenvectors** are given by

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \qquad X_n(x) = \sin\left(\frac{n\pi x}{a}\right),$$

where $n \geq 1$ integer.

Boundary conditions for Y

- We now turn to the associated ODE for $Y(y)$,

$$Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0. \tag{1}$$

- The boundary condition $u(x,b) = 0$, for $x \in [0,a]$, implies

$$X_n(x)Y(b) = 0, \quad \text{for } x \in [0,a].$$

$\left(Y(y) = c_1 e^{\frac{n\pi}{a}y} + c_2 e^{-\frac{n\pi}{a}y} \right)$

- Clearly, this means that $Y(b) = 0$.
- The general solution of ODE (1) can be written as

$$Y(y) = c_1 \cosh\left(\frac{n\pi(b-y)}{a}\right) + c_2 \sinh\left(\frac{n\pi(b-y)}{a}\right),$$

where c_1, c_2 arbitrary constants. (see next slide)

$0 = Y(b) = c_1 \cdot 1 + c_2 \cdot 0$
$= c_1$

- The condition $Y(b) = 0$ gives $c_1 = 0$. We thus pick

$$Y_n(y) = \sinh\left(\frac{n\pi(b-y)}{a}\right).$$

General solution of ODE for $Y(y)$

The **general solution** of the ODE for $Y(y)$ can be written as

$$Y(y) = c_1 \cosh\left(\frac{n\pi(b-y)}{a}\right) + c_2 \sinh\left(\frac{n\pi(b-y)}{a}\right),$$

where c_1, c_2 arbitrary constants.

$k = \frac{n\pi}{a}$, $Y_1(y) = \cosh(k(b-y))$, $Y_2(y) = \sinh(k(b-y))$.

Let's check that Y_1, Y_2 are solutions:

$$Y_1'(y) = \sinh(k(b-y)) \cdot (-k)$$

$$Y_1''(y) = \cosh(k(b-y)) \cdot (-k)^2$$

$$= Y_1(y) \cdot \left(\frac{n\pi}{a}\right)^2 \checkmark$$

$$Y_2'' = Y_2(y) \cdot \left(\frac{n\pi}{a}\right)^2 \checkmark$$

General solution of ODE for $Y(y)$ continued

To check linear independence, we compute their Wronskian:

$$W(Y_1, Y_2) = \begin{vmatrix} Y_1(y) & Y_2(y) \\ Y_1'(y) & Y_2'(y) \end{vmatrix} = \begin{vmatrix} \cosh(k(b-y)) & \sinh(k(b-y)) \\ (-k)\sinh(k(b-y)) & (-k)\cosh(k(b-y)) \end{vmatrix}$$

$$= (-k)\cosh(k(b-y))^2 - (-k)\sinh(k(b-y))^2$$

$$= -k\left(\cosh(k(b-y))^2 - \sinh(k(b-y))^2\right)$$

$$= -k \cdot 1 \neq 0$$

So $\{Y_1, Y_2\}$ form a fundamental set of solutions

Superposition

We conclude that, for every $n \geq 1$,

$$u_n(x, y) = X_n(x) Y_n(y) = \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right),$$

satisfies **Laplace's equation** as well as the **boundary conditions**

$$\begin{cases} u(0, y) = u(a, y) = 0 & \text{for } y \in (0, b), \\ u(x, b) = 0 & \text{for } x \in [0, a], \end{cases} \tag{2}$$

Note: Δ is a linear operator.

Applying the superposition principle

Through **superposition**, we obtain the following solution given by an infinite series,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right),$$

which still satisfies the three boundary conditions in equation (2).

The final boundary condition

Finally, we turn to the fourth **boundary condition** $u(x,0) = f(x)$, for $x \in [0,a]$, which now becomes

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = f(x), \quad \text{for } x \in [0,a].$$

Let's write

$$B_n \sinh\left(\frac{n\pi b}{a}\right) = C_n \qquad \text{for } n \geq 1,$$

then

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) = f(x), \quad \text{for } x \in [0,a].$$

This is a **Fourier sine series** with $L = a$. The **coefficients** are given by

$$C_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)\, dx \qquad \text{for } n \geq 1.$$

Complete solution to boundary value problem

The complete solution to the boundary value problem is given by

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right),$$

with **coefficients**

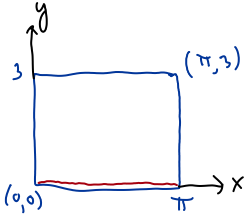
$$B_n = \frac{C_n}{\sinh\left(\frac{n\pi b}{a}\right)},$$

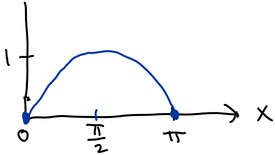
where

$$C_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)\, dx \qquad \text{for } n \geq 1.$$

Example

- We take a plate with length $a = \pi$ and width $b = 3$.
- We choose
$$f(x) = 1 - \left(\frac{2x}{\pi} - 1\right)^2.$$





Boundary value problem

Solve the following **boundary value problem** for **Laplace's equation** on a **rectangle**:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } (x,y) \in (0,\pi) \times (0,3), \\ u(0,y) = u(\pi,y) = 0 & \text{for } y \in (0,3), \\ u(x,3) = 0 & \text{for } x \in [0,\pi], \\ u(x,0) = f(x) & \text{for } x \in [0,\pi]. \end{cases}$$

Computing coefficients

The coefficients C_n, $n \geq 1$, of the **Fourier sine series**, are given by

$$
\begin{aligned}
C_n &= \frac{2}{a} \int_0^a \sin\left(\frac{n\,\pi\,x}{a}\right) f(x)\,dx \\
&= \frac{2}{\pi} \int_0^\pi \sin(n\,x) \left(1 - \left(\frac{2\,x}{\pi} - 1\right)^2\right) dx \\
&= \frac{2}{\pi} \int_0^\pi \sin(n\,x) \left(\frac{4\,x^2}{\pi^2} + \frac{4\,x}{\pi}\right) dx \\
&\vdots \\
&= \begin{cases} \frac{32}{\pi^3 n^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

Therefore

$$
B_n = \frac{C_n}{\sinh\left(\frac{n\,\pi\,b}{a}\right)} = \frac{C_n}{\sinh(3\,n)} = \begin{cases} \dfrac{32}{\sinh(3\,n)\,\pi^3 n^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
$$

The solution to the boundary value problem

The complete solution is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right), \\ &= \sum_{n=1}^{\infty} B_n \sin(n \,\text{[redacted]}\, x) \sinh(n(3-y)), \end{aligned}$$

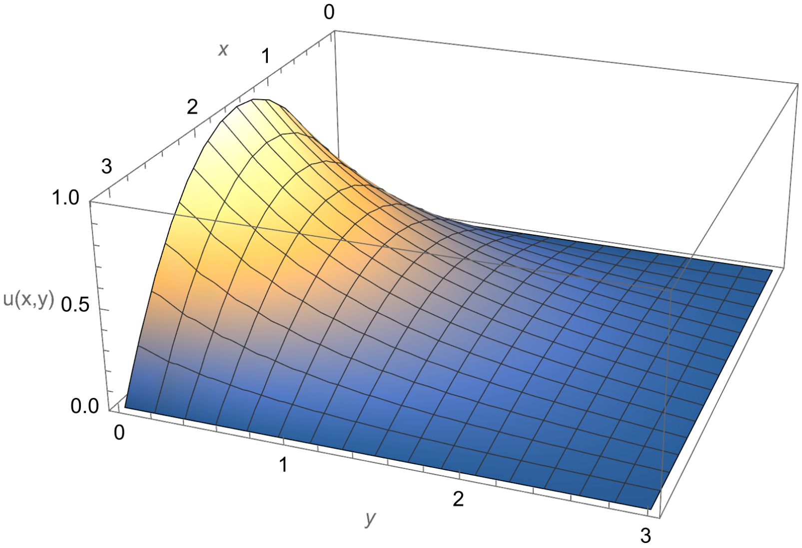
with

$$B_n = \begin{cases} \dfrac{32}{\sinh(3n)\,\pi^3 n^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

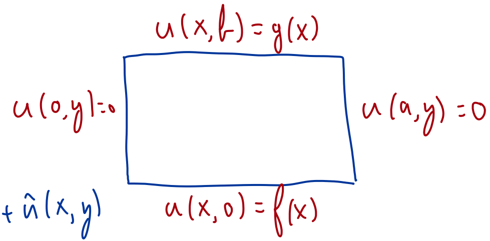
Writing $n = 2k+1$, this becomes

$$u(x,t) = \frac{32}{\pi^3} \sum_{k=0}^{\infty} \underbrace{\frac{1}{\sinh[6k+3](2k+1)^3}}_{B_{2k+1}} \sin[(2k+1) \,\text{[redacted]}\, x] \sinh[(2k+1)(3-y)].$$

Plot of solution

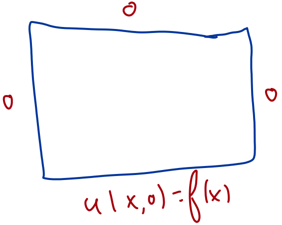


Additional slide

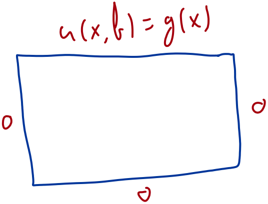


$u(x,y) = \tilde{u}(x,y) + \hat{u}(x,y)$

$\tilde{u}$:



$\hat{u}$:



summary

After today's lecture, you

- know how to solve boundary value problems for Laplace's equation on a rectangle.