MATH2021 - Differential Equations
Week 12, Lecture 2

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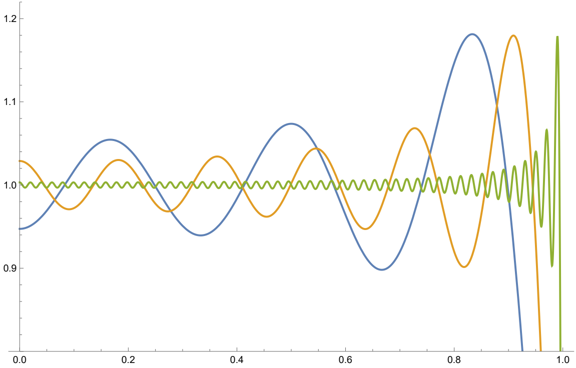
QUIZ 2: FRI
24th MAY



Past and present

- Last lecture, we
  - became familiar with partial differential equations (PDEs)
  - learnt the method of separation of variables
  - solved an initial-boundary value problem for the heat equation
- Today:
  - Heat equation for a bar with insulated ends
  - Heat equation with periodic boundary conditions

Recap: previous lecture

Initial-boundary value problem 1

For fixed $k, L > 0$, solve the following **initial-boundary value problem** for the **heat equation**:

$$\begin{cases} u_t = k\, u_{xx}, & \text{for } 0 < x < L, \quad t > 0, \\ u(0,t) = u(L,t) = 0, & \text{for } t > 0, \\ u(x,0) = f(x), & \text{for } 0 \le x \le L. \end{cases}$$

The heat equation with these boundary and initial conditions models the temperature distribution in an insulated bar of length L over time, where

- the initial temperature distribution at time $t = 0$ is given by $f(x)$,
- the temperatures at both ends of the bar, $x = 0$ and $x = L$, are kept at zero.

Solution of initial-boundary value problem

The solution to initial-boundary value problem 1 is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t},$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \text{for } n \geq 1.$$

Bar with insulated ends

- Rather than keeping the temperatures at the endpoints fixed at zero, we are going to assume that the ends are **insulated**.
- This means that there is **no heat flow** at $x = 0$ and at $x = L$.
- Mathematically,

$$u_x(0,t) = u_x(L,t) = 0 \qquad t > 0.$$

The corresponding initial-boundary value problem takes the following form.

Initial-boundary value problem 2

For fixed $k, L > 0$, solve the following **initial-boundary value problem** for the **heat equation**:

$$\begin{cases} u_t = k\, u_{xx}, & \text{for } 0 < x < L, \quad t > 0, \\ u_x(0,t) = u_x(L,t) = 0, & \text{for } t > 0, \\ u(x,0) = f(x), & \text{for } 0 \leq x \leq L. \end{cases}$$

The method of separation of variables

We look for solutions $u(x,t)$ that take the factorised form

$$u(x,t) = X(x)\,T(t).$$

Substitution into the heat equation $u_t = k\,u_{xx}$ gives

$$X(x)\,T'(t) = k\,X''(x)\,T(t).$$

We can rewrite this as

$$\lambda := -\frac{X''(x)}{X(x)} = -\frac{T'(t)}{k\,T(t)},$$

where λ is independent of x, t and thus a **constant**.

So we obtain the differential equations

$$X''(x) + \lambda\,X(x) = 0, \qquad T'(t) + \lambda\,k\,T(t) = 0.$$

The boundary conditions

The boundary conditions

$$u_x(0,t) = u_x(L,t) = 0, \qquad \text{for } t \geq 0,$$

imply

$$X'(0)\,T(t) = 0, \quad X'(L)\,T(t) = 0 \qquad \text{for } t > 0.$$

We do not want $T(t) \equiv 0$ and therefore we must impose

$$X'(0) = 0, \qquad X'(L) = 0.$$

We have obtained the following **eigenvalue problem**

$$X''(x) + \lambda\,X(x) = 0, \qquad X'(0) = 0, \quad X'(L) = 0,$$

with the accompanying differential equation

$$T'(t) + \lambda\,k\,T(t) = 0.$$

Solutions to eigenvalue problem

From Lecture 11-2, we know that the solutions of the eigenvalue problem are given by

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2,$$

for integers $n \geq 0$.

The general solution of the corresponding differential equation

$$T'(t) + \lambda_n k\, T(t) = 0,$$

is given by

$$T_n(t) = a_n\, e^{-\lambda_n k t},$$

where a_n an arbitrary constant.

Superposition

For every integer $n \geq 0$,

$$u_n(x,t) = X_n(x)\,T_n(t) = a_n \cos\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t},$$

is a solution to the **heat equation**

$$u_t = k\,u_{xx},$$

that satisfies the **boundary conditions**

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \qquad \text{for } t > 0.$$

Applying the superposition principle

Since the differential operator $\mathcal{L} : u \mapsto u_t - k\,u_{xx}$ is **linear**, we can use **superposition** to construct a more general solution as an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t},$$

which still satisfies the boundary conditions.

The initial condition

The **initial condition**

$$u(x,0) = f(x) \qquad \text{for } 0 \le x \le L,$$

reads

$$\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \qquad \text{for } 0 \le x \le L.$$

Recall from lecture 11-2 that the collection of functions

$$\left\{\cos\left(\frac{n\pi x}{L}\right) : n \ge 0 \text{ integer}\right\},$$

forms an **orthogonal family** on $[0, L]$.

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

for $m \neq n$, $m, n \geq 0$

Therefore

$$a_0 = \frac{\int_0^L f(x)\,dx}{\int_0^L 1\,dx} = \frac{1}{L}\int_0^L f(x)\,dx,$$

$$a_n = \frac{\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx}{\int_0^L \cos\left(\frac{n\pi x}{L}\right)^2\,dx} = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \quad (n \ge 1).$$

Fourier Cosine series

Fourier cosine series

The infinite series

$$\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

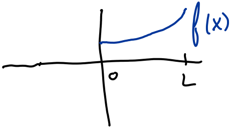
with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

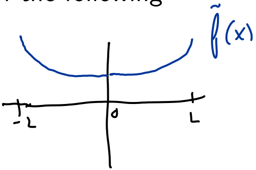
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad \text{for } n \geq 1,$$

is the **Fourier cosine series** of $f(x)$.

The Fourier cosine series of $f(x)$ is the Fourier series of the following **even** function on $[-L, L]$,



$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ f(-x) & \text{if } -L \leq x < 0 \end{cases}$$



Complete solution

The infinite series

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t},$$

with coefficients

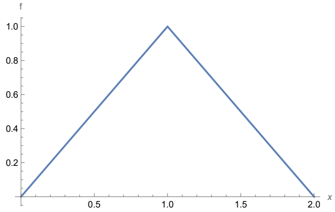
$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\,\pi\,x}{L}\right)dx \qquad \text{for } n \geq 1, \end{aligned}$$

is the solution to initial-boundary value problem 2.

Example

We set the parameter $k = \frac{1}{4}$, the length $L = 2$ and choose the initial temperature distribution at $t = 0$ as

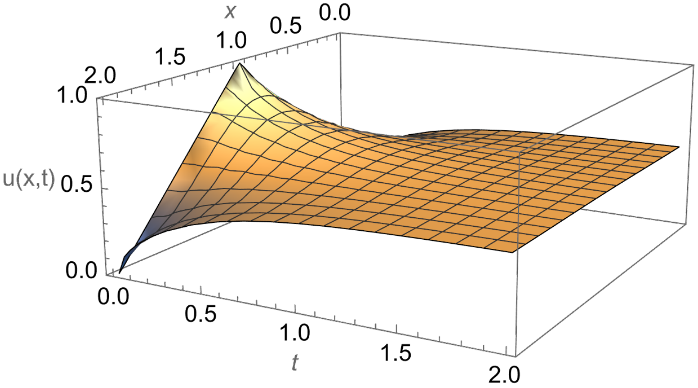
$$f(x) := 1 - |x - 1| \qquad \text{for } 0 \le x \le 2.$$



The solution to initial-boundary value problem 2 then becomes

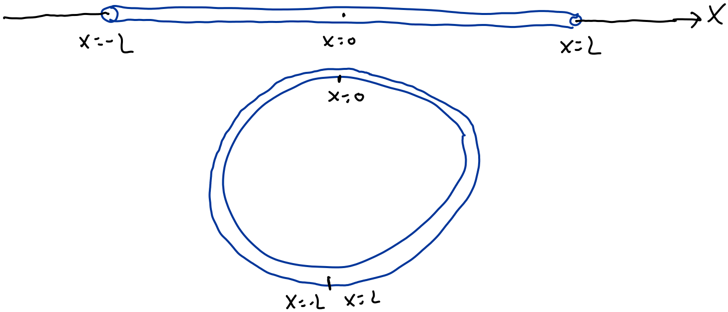
$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left(\frac{(2k+1)\pi x}{2}\right) \underbrace{e^{-\frac{1}{4}\left(\frac{(2k+1)\pi}{2}\right)^2 t}}_{\text{goes to } 0 \text{ as } t \to \infty}.$$

Numerical display of solution to IBV problem 2



Heat equation with periodic boundary conditions

Consider a rod of length $2L$ bend into a circle:



- The temperature must be the same at $x = -L$ and $x = L$,

$$u(-L, t) = u(L, t) \qquad t > 0.$$

- The heat flow must be the same at $x = -L$ and $x = L$,

$$u_x(-L, t) = u_x(L, t) \qquad t > 0.$$

IBV problem with periodic boundary conditions

Initial-boundary value problem 3

For fixed $k, L > 0$, solve the following **initial-boundary value problem** for the **heat equation**:

$$\begin{cases} u_t = k\, u_{xx}, & \text{for } -L < x < L, \quad t > 0, \\ u(-L,t) = u(L,t), & \text{for } t > 0, \\ u_x(-L,t) = u_x(L,t), & \text{for } t > 0, \\ u(x,0) = f(x), & \text{for } -L \le x \le L. \end{cases}$$

The boundary conditions

$$\begin{cases} u(-L,t) = u(L,t), & \text{for } t > 0, \\ u_x(-L,t) = u_x(L,t), & \text{for } t > 0, \end{cases}$$

are called **periodic boundary conditions**.

Separation of variables

We look for solutions $u(x,t)$ that take the factorised form

$$u(x,t) = X(x)\,T(t).$$

As before, substitution into the heat equation gives

$$X''(x) + \lambda X(x) = 0, \qquad T'(t) + \lambda\, k\, T(t) = 0.$$

for a **constant** λ.

This time, the boundary conditions become

$$\begin{aligned} X(-L)\,T(t) &= X(L)\,T(t), && \text{for } t > 0. \\ X'(-L)\,T(t) &= X'(L)\,T(t), && \text{for } t > 0. \end{aligned}$$

We do not want $T(t) \equiv 0$ and therefore we impose

$$\begin{aligned} X(-L) &= X(L), \\ X'(-L) &= X'(L). \end{aligned}$$

The eigenvalue problem

We have obtained the following **eigenvalue problem**

$$X''(x) + \lambda X(x) = 0, \qquad X(-L) = X(L), \quad X'(-L) = X'(L),$$

with the accompanying differential equation

$$T'(t) + \lambda k\, T(t) = 0.$$

From the theorem in Lecture 11-2, we know that the eigenvalue problem only admits non-negative eigenvalues.

Eigenvalue $\lambda = 0$

- Setting $\lambda = 0$, the general solution of the ODE for $X(x)$ is

$$X(x) = A + B\,x \qquad (A, B \text{ arbitrary constants}).$$

- The boundary condition $X(-L) = X(L)$ gives

$$A - B\,L = A + B\,L \implies 2\,B\,L = 0 \implies B = 0.$$

- Since $X'(x) = B$, the second boundary condition is trivially satisfied.

- So $X_0(x) = A_0$ is an **eigenfunction** with **eigenvalue** $\lambda_0 = 0$.

- We correspondingly pick $T_0(t) = 1$, so that

T ' = 0

$$u_0(x,t) = X_0(x)\,T_0(t) = A_0.$$

Eigenvalue $\lambda > 0$

- For $\lambda > 0$, we write $\lambda = \rho^2$, $\rho > 0$. $X''(x) + \rho^2 X(x) = 0$
- The general solution of the ODE for $X(x)$ is given by
$$X(x) = A\cos(\rho x) + B\sin(\rho x) \qquad (A, B \text{ arbitrary constants}).$$
- The derivative of $X(x)$ is given by
$$X'(x) = -A\rho\sin(\rho x) + B\rho\cos(\rho x).$$
- The **boundary condition** $X(-L) = X(L)$ gives
$$\underbrace{A\cos(\rho L) - B\sin(\rho L)}_{X(-L)} = A\cos(\rho L) + B\sin(\rho L)$$
$$\Downarrow$$
$$2B\sin(\rho L) = 0.$$
- The **boundary condition** $X'(-L) = X'(L)$ gives
$$A\rho\sin(\rho L) + B\rho\cos(\rho L) = -A\rho\sin(\rho L) + B\rho\cos(\rho L)$$
$$\Downarrow$$
$$\rho \cdot 2A\sin(\rho L) = 0.$$

Solving the eigenvalue problem (2/2)

- We have obtained two conditions,

$$2\,B\,\sin(\rho\,L) = 0, \qquad 2\,A\,\sin(\rho\,L) = 0.$$

- Since we do not want both $A = B = 0$, we require

$$\sin(\rho\,L) = 0.$$

$\lambda = \rho^2 = \left(\frac{n\pi}{L}\right)^2$

- It follows that $\rho\,L = n\,\pi$ for some $n \geq 1$.
- So, we have found **eigenvalues** and **eigenfunctions**

$$\lambda_n = \left(\frac{n\,\pi}{L}\right)^2, \quad X_n(x) = A_n \cos\left(\frac{n\,\pi\,x}{L}\right) + B_n \sin\left(\frac{n\,\pi\,x}{L}\right) \qquad n \geq 1.$$

$$\lambda_0 = 0, \qquad X_0(x) = A_0.$$

- For the corresponding ODE

$$T'(t) + \lambda_n\,k\,T(t) = 0,$$

we pick the solution

$$T_n(t) = e^{-\lambda_n\,k\,t}.$$

Superposition

$u_0(x,t) = A_0$, and

$$u_n(x,t) = X_n(x)\,T_n(t) = A_n\,\cos\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t} + B_n\,\sin\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t},$$

for $n \geq 1$, are solutions to the **heat equation**

$$u_t = k\,u_{xx},$$

satisfying the periodic boundary conditions.

Applying the superposition principle

Through **superposition**, we obtain the following solution given by an infinite series,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\,\cos\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t} + B_n\,\sin\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t},$$

which still satisfies the periodic boundary conditions.

The initial condition

The **initial condition**

$$u(x,0) = f(x) \qquad \text{for } -L \le x \le L,$$

reads

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \qquad \text{for } -L \le x \le L.$$

This means that the A_n and B_n are the standard Fourier coefficients,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$
$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (n \ge 1).$$

Complete solution

The infinite series

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t} + B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t},$$

with coefficients

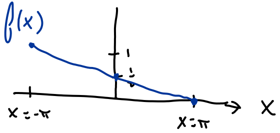
$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$
$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (n \geq 1).$$

is the solution to initial-boundary value problem 3.

Example

We set the parameter $k = \frac{1}{10}$, the length $L = \pi$ and choose the initial temperature distribution at $t = 0$ as

$$f(x) := \frac{1}{2} - \frac{x}{2\pi} \qquad \text{for } -\pi \le x \le \pi.$$



Initial-boundary value problem 4

Solve the following **initial-boundary value problem** for the **heat equation**:

$$\begin{cases} u_t = \frac{1}{10}\, u_{xx}, & \text{for } -\pi < x < \pi, \quad t > 0, \\ u(-\pi, t) = u(\pi, t), & \text{for } t > 0, \\ u_x(-\pi, t) = u_x(\pi, t), & \text{for } t > 0, \\ u(x, 0) = \frac{1}{2} - \frac{x}{2\pi}, & \text{for } -\pi \le x \le \pi. \end{cases}$$

Solution

$f(x) = \frac{1}{2} - \frac{x}{2\pi}$

$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2} - \frac{x}{2\pi}\,dx = \frac{1}{2\pi}\left(2\pi\cdot\frac{1}{2} + 0\right)$
$= \frac{1}{2}$

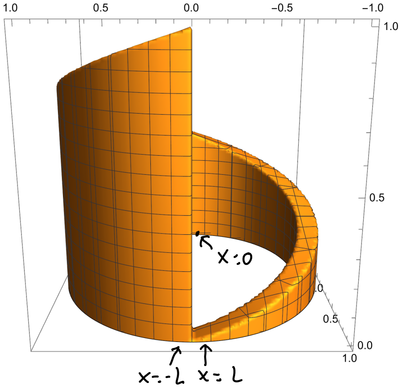
$A_n = 0 \quad n \geq 1.$

$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} - \frac{x}{2\pi}\right)\sin(nx)\,dx$
$= \frac{(-1)^n}{\pi\cdot n}$

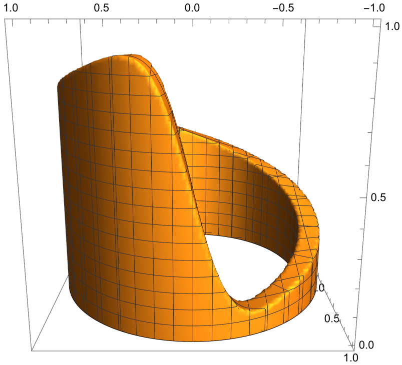
Solution to initial-boundary value problem 4:

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi\, n}\sin(n\,x)\; e^{-\frac{1}{10}n^2\,t}.$$

Temperature plots

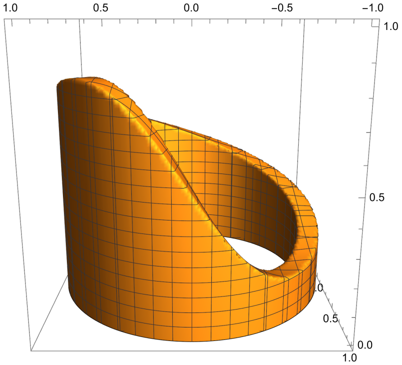


$t = 0$

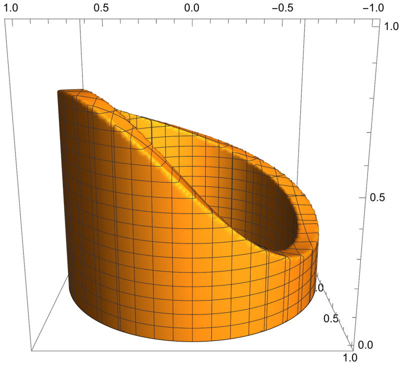


$t = 0.01$

Temperature plots

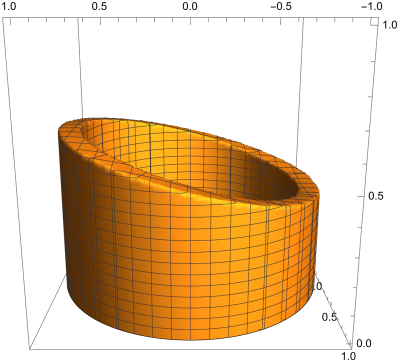


$t = 0.1$

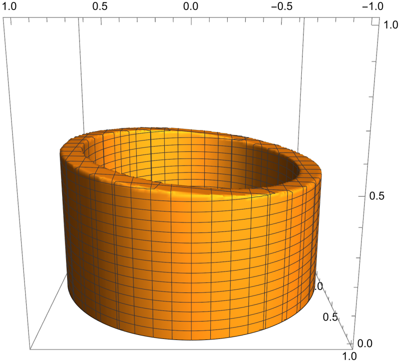


$t = 0.2$

Temperature plots



$t = 1$



$t = 2$

summary

After today's lecture, you know how to solve initial-boundary value problems for the heat equation with various boundary conditions.