MATH2021 - Differential Equations
Week 12, Lecture 1

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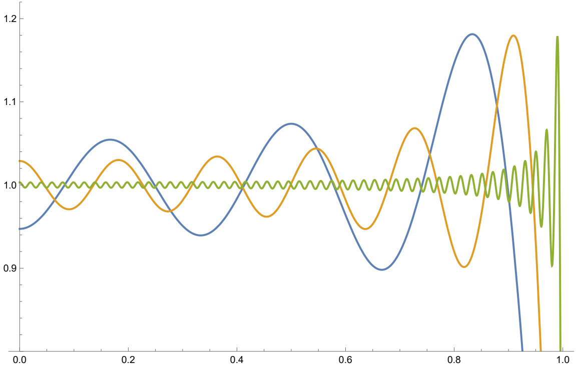


Past and present

- Last lecture:
  - periodic funtions
  - Fourier series
  - Fourier convergence theorem
- Today:
  - Partial differential equations
  - The heat equation
  - Fourier sine series

Differential Equations

Ordinary Differential Equations (ODEs)

An **ordinary differential equation** is an equation involving an unknown function $y(x)$, as well as one or more of its derivatives $y'(x), y''(x), y'''(x), \ldots,$ and possible the variable x.

Partial Differential Equations (PDEs)

A **partial differential equation** is an equation involving:
- an unknown function $u(x_1, x_2, \ldots, x_n)$ depending on $n \geq 2$ variables,
- one or more of its **partial derivatives**

$$
\begin{array}{llllllr}
u_{x_1}, & u_{x_2}, & u_{x_3}, & \ldots & u_{x_{n-1}}, & u_{x_n}, & \text{(1st order)} \\
u_{x_1x_1}, & u_{x_1x_2}, & u_{x_1x_3}, & \ldots & u_{x_{n-1}x_n}, & u_{x_nx_n}, & \text{(2nd order)} \\
u_{x_1x_1x_1}, & u_{x_1x_1x_2}, & u_{x_1x_1x_3}, & \ldots & u_{x_{n-1}x_nx_n}, & u_{x_nx_nx_n}, & \text{(3th order)} \\
u_{x_1x_1x_1x_1}, & u_{x_1x_1x_1x_2}, & u_{x_1x_1x_1x_3}, & \ldots & u_{x_{n-1}x_nx_nx_n}, & u_{x_nx_nx_nx_n}, & \text{(4th order)} \\
\vdots & \vdots & \vdots & & \vdots & \vdots &
\end{array}
$$

and possibly some of the variables $x_1, x_2, \ldots, x_n$.

Examples of PDEs

Name:	**unknown function:**	**PDE:**
(1-dim.) heat equation	$u(x,t)$	$u_t = k\, u_{xx}$ (k is a parameter)
(2-dim.) Laplace equation	$u(x,y)$	$u_{xx} + u_{yy} = 0$
(1-dim.) wave equation	$u(x,t)$	$u_{tt} = a^2 u_{xx}$
(2-dim.) wave equation	$u(x,y,t)$	$u_{tt} = a^2 (u_{xx} + u_{yy})$
Burgers' equation	$u(x,t)$	$u_t + u\, u_x = \nu u_{xx}$
Korteweg–De Vries equation	$u(x,t)$	$u_t + u_{xxx} - 6\, u\, u_x = 0$
Sine-Gorden equation	$\varphi(x,t)$	$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0$

Example of a solution

Show that $u(x,t) = e^{-8t} \sin 2x$ is a **solution** of the heat equation

$$u_t = 2\,u_{xx}.$$

$$u_t = -8\, e^{-8t} \sin 2x$$

$$u_x = 2 e^{-8t} \cos 2x$$

$$u_{xx} = -4\, e^{-8t} \sin 2x$$

Substitution into the PDE gives:

$$LHS = -8\, e^{-8t} \sin 2x$$

$$RHS = -8\, e^{-8t} \sin 2x$$

LHS = RHS ✓

Comparison of general solutions for ODEs and PDEs

The ODE $y'(x) = 2$ has general solution

$$y(x) = 2x + c,$$

where c an **arbitrary constant**.

Consider now the PDE for $u = u(x, y)$,

$$u_x = \cos y.$$

Integrating both sides with respect to x, gives

$$u(x, y) = x \cos y + h(y),$$

where $h(y)$ is an **arbitrary function**.

General solutions of PDEs contain arbitrary functions.

Heat equation with boundary and initial conditions

Let $k > 0$ and consider the heat equation

$$u_t = k\, u_{xx}.$$

We fix an $L > 0$ and look for a solution $u(x,t)$ on the domain

$$\{(x,t) : 0 \le x \le L \text{ and } t \ge 0\} = [0,L] \times [0,\infty),$$

that satisfies the **boundary conditions**

$$u(0,t) = 0, \quad u(L,t) = 0, \qquad \text{for } t \ge 0,$$

and the **initial condition**

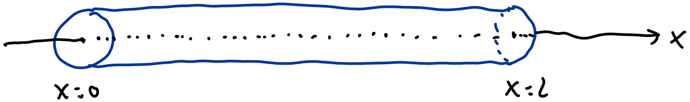
$$u(x,0) = f(x) \qquad \text{for } 0 \le x \le L.$$

- Boundary conditions refer to conditions at the boundary in the spatial direction.
- Initial conditions refer to conditions at an initial time $t = t_0$.
- A partial differential equation together with boundary$\overset{+}{/}$initial conditions is called [redacted]

AN INITIAL-BOUNDARY VALUE PROBLEM.

Physical interpretation

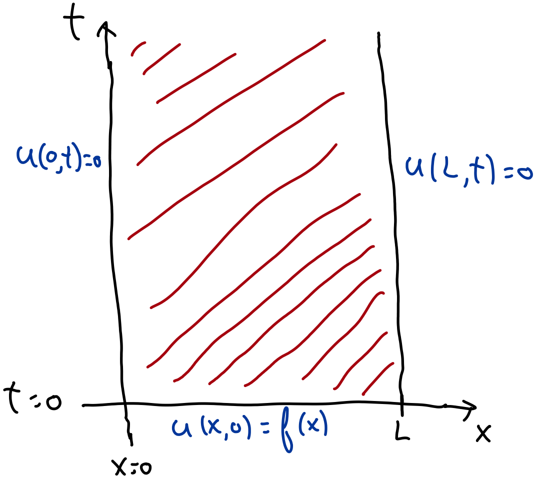
The heat equation with these boundary and initial conditions models the temperature distribution in an insulated bar of length L,



over time, where

- the temperatures at both ends of the bar, $x = 0$ and $x = L$, are kept at zero,
- the initial temperature distribution at time $t = 0$ is given by $f(x)$,
- the parameter k describes the thermal conductivity of the bar.

Drawing of boundary and initial conditions



The method of separation of variables

The method of **separation of variables** involves looking for solutions $u(x,t)$ that take the factorised form

$$u(x,t) = X(x)\,T(t).$$

Note that

$$\begin{aligned} u_t(x,t) &= X(x)\,T'(t), \\ u_x(x,t) &= X'(x)\,T(t), \\ u_{xx}(x,t) &= X''(x)\,T(t). \end{aligned}$$

Substitution into the heat equation $u_t = k\,u_{xx}$ gives

$$X(x)\,T'(t) = k\,X''(x)\,T(t).$$

divide by $k X(x) T(t)$

We can rewrite this as

$$\boxed{\frac{X''(x)}{X(x)} = \frac{T'(t)}{k\,T(t)}.}$$

The method of separation of variables continued

Let us write

$$\lambda := -\frac{X''(x)}{X(x)} \tag{1}$$

$$= -\frac{T'(t)}{k\,T(t)}. \tag{2}$$

- By equation (1), λ does not depend on t.
- By equation (2), λ does not depend on x.
- So λ is a **constant**.
- We can now rewrite equations (1) and (2) as

$$X''(x) + \lambda X(x) = 0, \qquad T'(t) + \lambda k\,T(t) = 0.$$

The boundary conditions

The boundary conditions

$u(x,t) = X(x) \cdot T(t)$

$$u(0,t) = u(L,t) = 0, \qquad \text{for } t \geq 0,$$

imply

$$X(0)\,T(t) = 0, \quad X(L)\,T(t) = 0 \qquad \text{for } t \geq 0.$$

We do not want $T(t) \equiv 0$ and therefore we must impose

$$X(0) = 0, \qquad X(L) = 0.$$

We have obtained the following **eigenvalue problem**

$$X''(x) + \lambda\, X(x) = 0, \qquad X(0) = 0, \quad X(L) = 0,$$

with the accompanying differential equation

$$T'(t) + \lambda\, k\, T(t) = 0.$$

Solutions to eigenvalue problem

From Lecture 11-2, we know that the solutions of the eigenvalue problem are given by

$$X_n(x) = \sin\left(\frac{n\,\pi\,x}{L}\right), \qquad \lambda_n = \left(\frac{n\,\pi}{L}\right)^2,$$

for integers $n \geq 1$.
The general solution of the corresponding differential equation

$$T'(t) + \lambda_n\, k\, T(t) = 0,$$

is given by

$$T_n(t) = b_n\, e^{-\lambda_n\, k\, t},$$

where b_n an arbitrary constant.

Superposition

For every integer $n \geq 1$,

$$u_n(x,t) = X_n(x)\,T_n(\overset{t}{\cancel{x}}) = b_n \sin\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t},$$

is a solution to the **heat equation**

$$u_t = k\,u_{xx},$$

that satisfies the **boundary conditions**

$$u(0,t) = 0, \quad u(L,t) = 0, \qquad \text{for } t \geq 0.$$

Applying the superposition principle

Since the differential operator $\mathcal{L} : u \mapsto u_t - k\,u_{xx}$ is **linear**, we can use **superposition** to construct a more general solution as an infinite series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\,\pi\,x}{L}\right) e^{-\left(\frac{n\,\pi}{L}\right)^2 k\,t},$$

which still satisfies the boundary conditions.

The initial condition

The **initial condition**

$$u(x,0) = f(x) \qquad \text{for } 0 \le x \le L,$$

reads

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \qquad \text{for } 0 \le x \le L.$$

Recall from lecture 11-2 that the collection of functions

$$\left\{ \sin\left(\frac{n\pi x}{L}\right) : n \ge 1 \text{ integer} \right\},$$

forms an **orthogonal family** on $[0, L]$.

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0$$

for $m \neq n$, $m, n \geq 1$.

Therefore

$$b_n = \frac{\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx}{\int_0^L \sin\left(\frac{n\pi x}{L}\right)^2 dx} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$\frac{L}{2}$

Fourier Sine series

Fourier sine series

The series

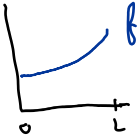
$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

with coefficients

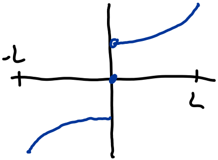
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \text{for } n \geq 1,$$

is the **Fourier sine series** of $f(x)$.

The Fourier sine series of $f(x)$ is the Fourier series of the following **odd** function on $[-L, L]$,



$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 < x \leq L \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } -L \leq x < 0 \end{cases}$$



Complete solution

The infinite series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k\,t},$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \text{for } n \geq 1,$$

is a solution to the heat equation

$$u_t = k\,u_{xx},$$

that satisfies the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \qquad \text{for } t \geq 0,$$

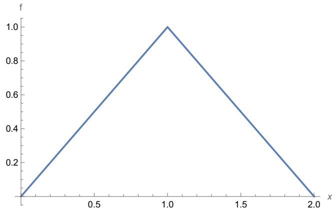
and the initial condition

$$u(x,0) = f(x) \qquad \text{for } 0 \leq x \leq L.$$

Example

We set the parameter $k = \frac{1}{4}$, the length $L = 2$ and choose the initial temperature distribution at $t = 0$ as

$$f(x) := 1 - |x - 1| \qquad \text{for } 0 \leq x \leq 2.$$



INITIAL-Boundary value problem

Find the solution to the heat equation

$$u_t = \frac{1}{4}\, u_{xx},$$

satisfying the boundary conditions and initial condition

$$u(0, t) = 0, \quad u(2, t) = 0, \qquad \text{for } t \geq 0,$$

$$u(x, 0) = 1 - |x - 1| \quad \text{for } 0 \leq x \leq 2.$$

Fourier sine series

$f(x) = 1 - |x-1|$

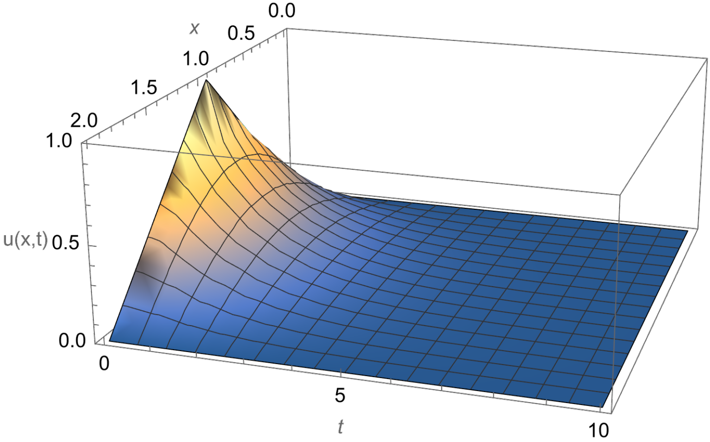
The Fourier sine series of $f(x)$ is given by

$$f(x) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left(\frac{(2k+1)\pi x}{2}\right).$$

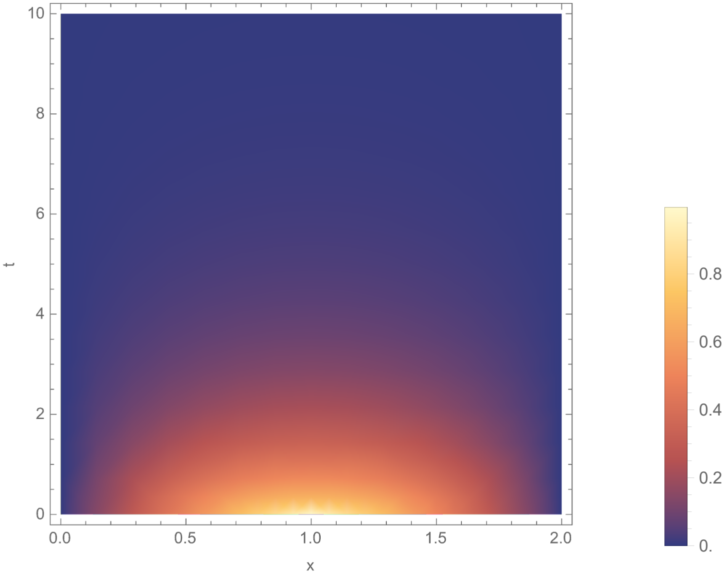
The solution to the INITIAL- boundary value problem is given by

$$u(x,t) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left(\frac{(2k+1)\pi x}{2}\right) e^{-\frac{1}{4}\left(\frac{(2k+1)\pi}{2}\right)^2 t}.$$

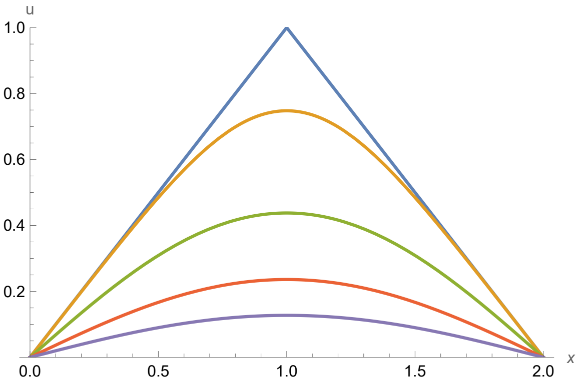
Numerical display of solution 1



Numerical display of solution 2



Numerical display of solution 3



- $u(x,0) = f(x)$ in blue
- $u(x,0.2)$ in orange
- $u(x,1)$ in green
- $u(x,2)$ in red
- $u(x,3)$ in purple

summary

After today's lecture, you

- know how to apply the **method of separation of variables** to the heat equation.
- be able to compute the **Fourier sine series** of a function.
- be able to solve the **heat equation** with zero boundary conditions and any initial condition.