MATH2021 - Differential Equations
Week 11, Lecture 3

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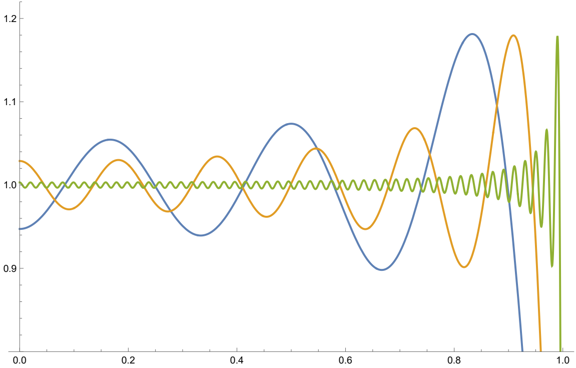


Past and present

- Last lecture was about
  - eigenvalue problems
  - orthogonal families of functions
- Today is about
  - periodic functions
  - Fourier series
  - the Fourier convergence theorem
  - partial Fourier sums

Fourier series

We finished the last lecture showing the following.

Suppose a function $f(x)$ on $[-L, L]$ can be written as a series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \tag{1}$$

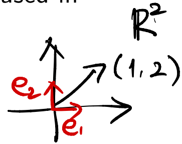
where we assume the series on the right-hand side converges nicely.
Then the **coefficients** are given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (n \geq 1).$$

- The right-hand side of equation (1) is called a **Fourier series**.
- It is natural to ask what kind of functions can be written as in (1).
- We find an answer to this question today.

Background

- Joseph Fourier first used **Fourier series** to find solutions to the **heat equation**. (More on this in Lecture 12-1)
- **Fourier series** are widely used in physics and engineering.
- Applications include
  - acoustics
  - electrical engineering
  - optics
  - quantum mechanics
  - signal processing
  - ⋮



Periodic functions

A function $f(x)$ is called **periodic** if there exists a $T > 0$ such that

$$f(x+T) = f(x) \qquad \text{for all } x.$$

In that case, T is called a **period** of the function.

Examples:

- $f(x) = \cos(x)$ is periodic with period $T = 2\pi$.
- $f(x) = \sin\left(\frac{n\pi x}{L}\right)$ is periodic with period $T = 2L$, for $n \in \mathbb{Z}$.
- $f(x) = x^2$ is not periodic.
- $f(x) = 1$ is periodic with period any $T > 0$.
- The Fourier series

$$\mathcal{F}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

  is periodic with period $T = 2L$.

The Fourier series of a function

Fix an $L > 0$ and let $f(x)$ be a **periodic function** with period $2L$.

Then the **Fourier series** of $f(x)$ is defined by

$$\mathcal{F}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

with coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (n \geq 1).$$

For $N \geq 0$, the Nth partial Fourier sum of $f(x)$ is given by

$$\mathcal{F}_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

Piecewise continuously differentiable functions

A function $f(x)$ on an interval $[a, b]$ is called **piecewise continuously differentiable**, if we can chop up the interval into finitely many subintervals,

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_{n-1}, x_n],$$

with

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b,$$

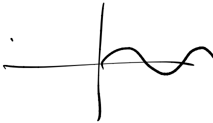
such that $f(x)$ and $f'(x)$ are continuous on each open subinterval (x_k, x_{k+1}) and the following limits exist,

$$\lim_{x \to x_k^+} f(x), \qquad \lim_{x \to x_k^+} f'(x), \qquad \lim_{x \to x_{k+1}^-} f(x), \qquad \lim_{x \to x_{k+1}^-} f'(x),$$

for $0 \leq k \leq n - 1$.

A function $f : \mathbb{R} \to \mathbb{R}$ is called **piecewise continuously differentiable**, if it is piecewise continuously differentiable on every interval $[a, b] \subseteq \mathbb{R}$.

① continuously diff-able on [a,b].



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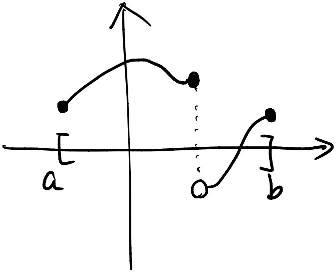
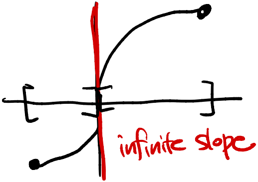


piecewise continuously diff-able

③



Not piecewise continuously diff-able.

Fourier convergence theorem

The Fourier convergence theorem

Fix $L > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuously differentiable function that is periodic with period $T = 2L$.

Then its Fourier series

$$\mathcal{F}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

converges for every $x \in \mathbb{R}$.

Furthermore, for any $x_0 \in \mathbb{R}$,

- If $f(x)$ is continuous at $x = x_0$, then

$$\mathcal{F}(x_0) = f(x_0).$$

- If $f(x)$ is not continuous at $x = x_0$, then

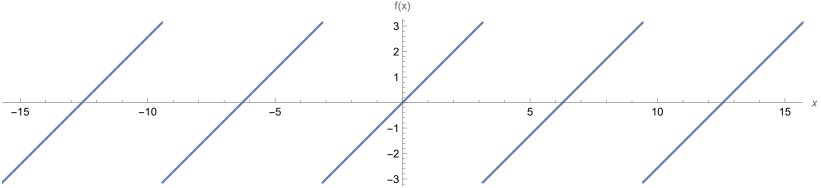
$$\mathcal{F}(x_0) = \frac{1}{2}\Big(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\Big).$$

Example: the sawtooth wave

Define

$$f(x) = \begin{cases} x & \text{if } -\pi \le x < \pi, \\ f(x+2\pi) & \text{for all } x \in \mathbb{R}. \end{cases}$$

This function is periodic with period $T = 2\pi$.



Let's compute its Fourier series with $L = \pi$,

$$\mathcal{F}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

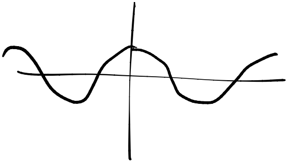
Computing the a_n's

odd $f(-x) = -f(x)$.

The constant coefficient is given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0,$$

because x is an odd function.



$fg = f(x)g(x)$

Similarly,

$fg(-x) \stackrel{?}{=} -fg(x)$.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0,$$

because $x\cos(nx)$ is an odd function.

even : $f(-x) = f(x)$.

Computing the b_n's

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\frac{1}{L}\cdot 2\cdot\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left(\left[-x\frac{\cos(nx)}{n}\right]_{x=-\pi}^{x=\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}\,dx\right)$$

$$= \frac{1}{\pi}\left(-\pi\frac{\cos(n\pi)}{n} - \left(-(-\pi)\frac{\cos(-n\pi)}{n}\right) + \left[\frac{\sin(nx)}{n^2}\right]_{x=-\pi}^{x=\pi}\right)$$

$$= \frac{1}{\pi}\left(-2\pi\frac{\cos(n\pi)}{n} + \frac{\sin(n\pi)}{n^2} - \frac{\sin(-n\pi)}{n^2}\right)$$

$$= \frac{1}{\pi}\left(-2\pi\frac{(-1)^n}{n} + \frac{0}{n^2} - \frac{0}{n^2}\right) = (-1)^{n+1}\frac{2}{n}$$

$n=0: \cos 0 = 1$

$n=1: \cos(\pi) = -1$

$n=2: \cos(2\pi) = 1$

Partial Fourier sums

The Fourier series of $f(x)$ is given by

$$\begin{aligned}\mathcal{F}(x) &= 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin(nx)\\ &= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right)\end{aligned}$$

Some of the first few **partial Fourier sums** are given by

1st partial Fourier sum: $\mathcal{F}_1(x) = 2\sin x,$

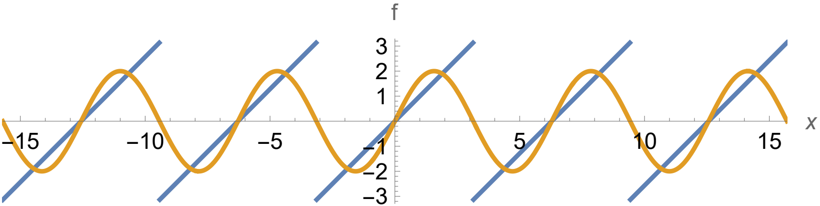
2nd partial Fourier sum: $\mathcal{F}_2(x) = 2\left(\sin x - \frac{1}{2}\sin 2x\right),$

3th partial Fourier sum: $\mathcal{F}_3(x) = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x\right),$

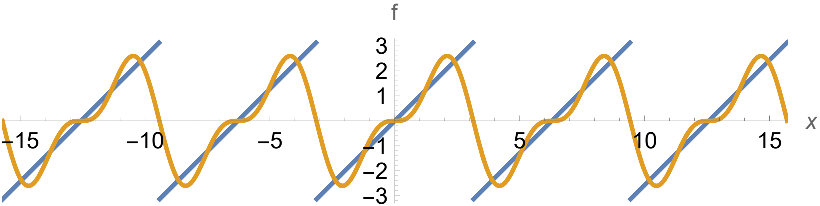
4th partial Fourier sum: $\mathcal{F}_4(x) = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x\right).$

Plots of partial Fourier sums 1, 2

1st partial sum $\mathcal{F}_1(x)$ in orange:

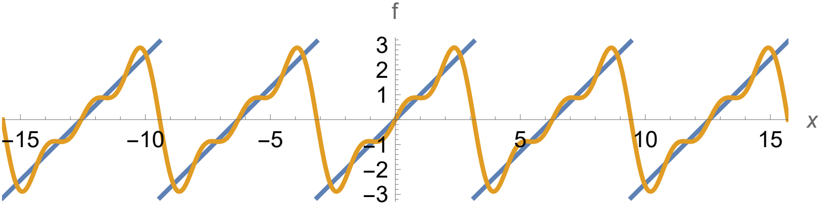


2nd partial sum $\mathcal{F}_2(x)$ in orange:

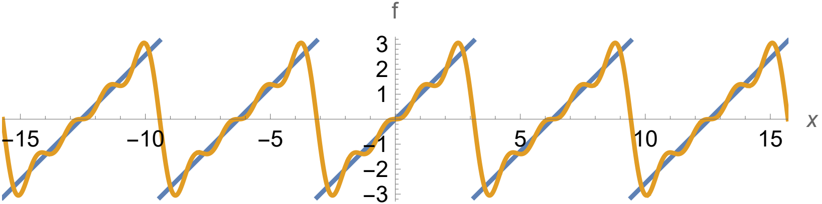


Plots of partial Fourier sums 3, 4

3th partial sum $\mathcal{F}_3(x)$ in orange:

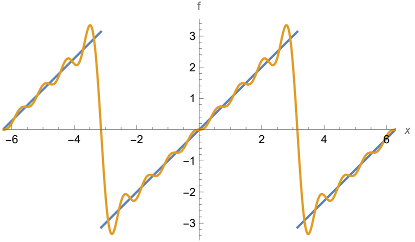


4th partial sum $\mathcal{F}_4(x)$ in orange:

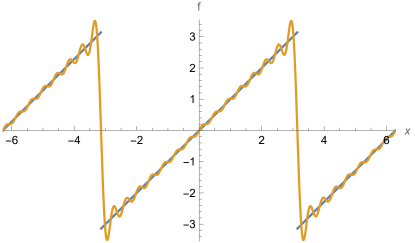


Plots of partial Fourier sums 8, 15

8th partial sum $\mathcal{F}_8(x)$ in orange:



15th partial sum $\mathcal{F}_{15}(x)$ in orange:



Reflections on the plots

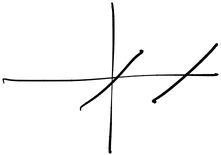
In accordance with the **Fourier convergence theorem**:

- At points $x_0 \neq \pi(2n+1)$, for all $n \in \mathbb{Z}$, the function $f(x)$ is continuous and we see that

$$\lim_{N\to\infty} \mathcal{F}_N(x_0) = x_0 = f(x_0).$$

- At points $x_0 = \pi(2n+1)$, for some $n \in \mathbb{Z}$, the function $f(x)$ is not continuous and we see that

$$\lim_{N\to\infty} \mathcal{F}_N(x_0) = 0 = \frac{1}{2}(1 + (-1)) = \frac{1}{2}\Big(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\Big).$$

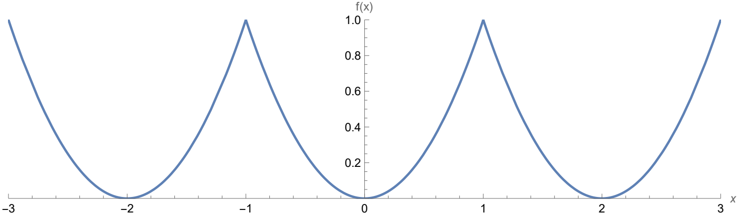


Periodic extension

- Sometimes we want to write functions that are **not periodic** as Fourier series.
- Take for example the function $f(x) = x^2$.
- We can turn $f(x)$ into a periodic function, with period 2, by defining

$$\tilde{f}(x) = \begin{cases} x^2 & \text{if } -1 \le x < 1, \\ \tilde{f}(x+2) & \text{for all } x \in \mathbb{R}. \end{cases}$$

  This is called the 2-**periodic extension** of the function $f(x) = x^2$ on $[-1, 1]$.



Periodic extensions and Fourier series

- In general, let $L > 0$ and given a function $f(x)$ defined on the interval $[-L, L]$, we define the $2L$-periodic extension of $f(x)$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } -L < x < L, \\ \frac{1}{2}(f(-L) + f(L)) & \text{if } x = L, \\ \tilde{f}(x + 2L) & \text{for all } x \in \mathbb{R}. \end{cases}$$

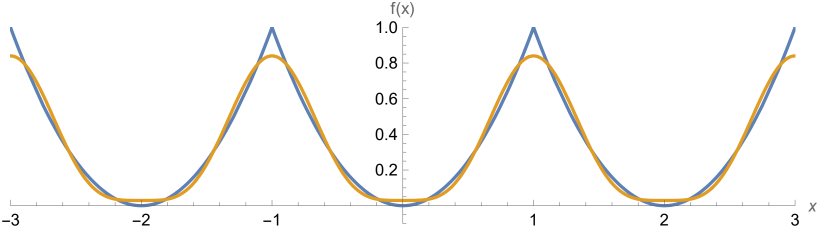
- The Fourier series of $f(x)$ on $[-L, L]$ is by definition the Fourier series of its $2L$-periodic extension $\tilde{f}(x)$.

- The Fourier convergence theorem tells us that the Fourier series of $f(x)$ will converge to $f(x)$ at all points $x_0 \in (-L, L)$ where $f(x)$ is continuous.

summary

After today's lecture, you

- know how to compute the Fourier series of a function,
- understand the Fourier convergence theorem,
- know what periodic extensions of functions are and how to sketch them.



The 2nd partial Fourier sum of the 2-periodic extension of x^2 in orange.