MATH2021 - Differential Equations
Week 11, Lecture 2

CONSULTATION IS TODAY 1-2 PM

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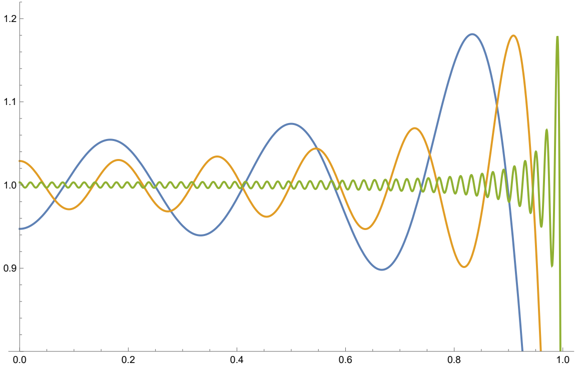


Past and present

- Last lecture:
  - using the Laplace transform to solve initial value problems.
- Today:
  - eigenvalue problems
  - orthogonal families
  - series of functions

Eigenvalue problems

Eigenvalue problem

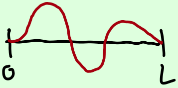
Fix an $L > 0$ and define the differential operator $\mathcal{L}[y] = -y''$.
The differential equation

$$\mathcal{L}[y] = \lambda y,$$

linear algebra:
$A \cdot v = \lambda \cdot v$

with **boundary conditions**

$$y(0) = 0, \quad y(L) = 0,$$



defines an **eigenvalue problem**.

- A value of $\lambda \in \mathbb{R}$, for which there exists a non-trivial solution $y(x)$, is called an **eigenvalue**.
- The corresponding solution is called an **eigenfunction**.
- The problem consists of finding all pairs of eigenvalues and eigenfunctions.
- Eigenvalue problems will naturally come up when we study **partial differential equations** in weeks 12 and 13.

Solving the eigenvalue problem

The differential equation $\mathcal{L}[y] = \lambda y$ can be written as

$$y'' + \lambda y = 0.$$

The corresponding **characteristic polynomial** is given by

$$P(r) = r^2 + \lambda.$$

The nature of solutions depends whether λ is positive, zero or negative.

We thus look at these three cases individually:

(1) $\lambda < 0$,

(2) $\lambda = 0$,

(3) $\lambda > 0$.

Case (1): $\lambda < 0$

We may write $\lambda = -k^2$, for some $k > 0$. The ODE becomes

$$y'' - k^2 y = 0,$$

with general solution

$$y(x) = c_1 e^{kx} + c_2 e^{-kx}.$$

Imposing the boundary conditions gives

$$\begin{aligned} y(0) = 0 \qquad &\Longrightarrow \qquad c_1 + c_2 = 0, \\ y(L) = 0 \qquad &\Longrightarrow \qquad c_1 e^{kL} + c_2 e^{-kL} = 0. \end{aligned}$$

The linear system on the right has only the trivial solution $c_1 = c_2 = 0$, because

$$\begin{vmatrix} 1 & 1 \\ e^{kL} & e^{-kL} \end{vmatrix} = e^{-kL} - e^{kL} = e^{-kL}(1 - e^{2kL}) \neq 0.$$

So, there are no negative eigenvalues.

Case (2): $\lambda = 0$

The ODE is given by

$$y'' = 0,$$

with general solution

$$y(x) = c_1 + c_2\,x.$$

Imposing the boundary conditions gives

$$\begin{aligned} y(0) = 0 &\qquad \Longrightarrow \qquad & c_1 = 0, \\ y(L) = 0 &\qquad \Longrightarrow \qquad & c_1 + c_2\,L = 0. \end{aligned}$$

The linear system on the right has only the trivial solution $c_1 = c_2 = 0$.

So, $\lambda = 0$ is not an eigenvalue.

Case (3): $\lambda > 0$

We may write $\lambda = k^2$, for some $k > 0$. The ODE becomes

$$y'' + k^2 y = 0,$$

with general solution

$$y(x) = c_1 \cos(k\,x) + c_2 \sin(k\,x).$$

Imposing the boundary conditions gives

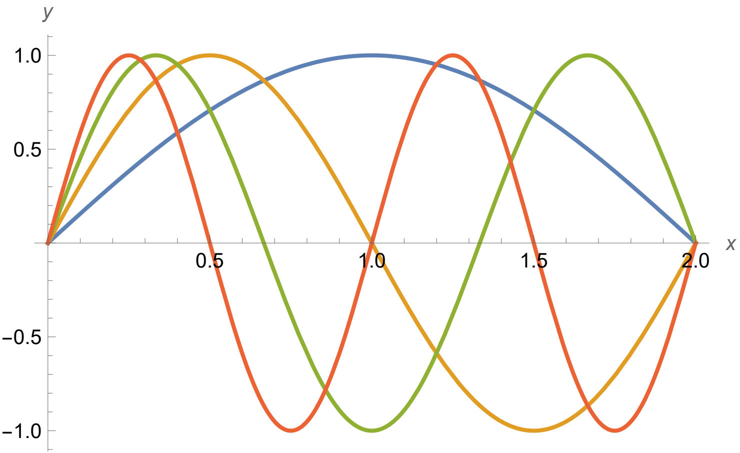
$$\begin{aligned} y(0) = 0 \quad &\Longrightarrow \quad c_1 = 0 &&\Longrightarrow \quad y(x) = c_2 \sin(k\,x) \\ y(L) = 0 \quad &\Longrightarrow \quad c_2 \sin(k\,L) = 0. &&\Longrightarrow \quad k \cdot L = \pi \cdot n, \text{ for some } n \in \mathbb{Z}_{>0} \end{aligned}$$

For every integer $n \geq 1$, we obtain a solution

$$k_n = \frac{n\,\pi}{L}, \quad \lambda_n = \left(\frac{n\,\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\,\pi\,x}{L}\right).$$

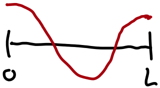
These are all the solutions to the eigenvalue problem.

Plots of first few eigenfunctions with $L = 2$



- $y_1(x)$ in blue
- $y_2(x)$ in orange
- $y_3(x)$ in green
- $y_4(x)$ in red

Eigenvalue problem with different boundary conditions



Another eigenvalue problem:

Eigenvalue problem

Fix an $L > 0$ and consider the **eigenvalue problem**

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(L) = 0.$$

- There are again no negative eigenvalues.
- Setting $\lambda = 0$, the general solution of the ODE is $y(x) = c_1 + c_2 x$. Imposing the boundary conditions gives $y'(x) = c_2$

$$\begin{aligned} y'(0) = 0 \qquad &\Longrightarrow \qquad c_2 = 0, \\ y'(L) = 0 \qquad &\Longrightarrow \qquad c_2 = 0. \end{aligned}$$

This time $\lambda_0 = 0$ is an eigenvalue, with eigenfunction $y_0(x) = 1$.

Positive eigenvalues

- For $\lambda = k^2$, $k > 0$, the general solution is

$$y(x) = c_1 \cos(k\,x) + c_2 \sin(k\,x).$$

- Its derivative is given by

$$y'(x) = -k\,c_1 \sin(k\,x) + k\,c_2 \cos(k\,x).$$

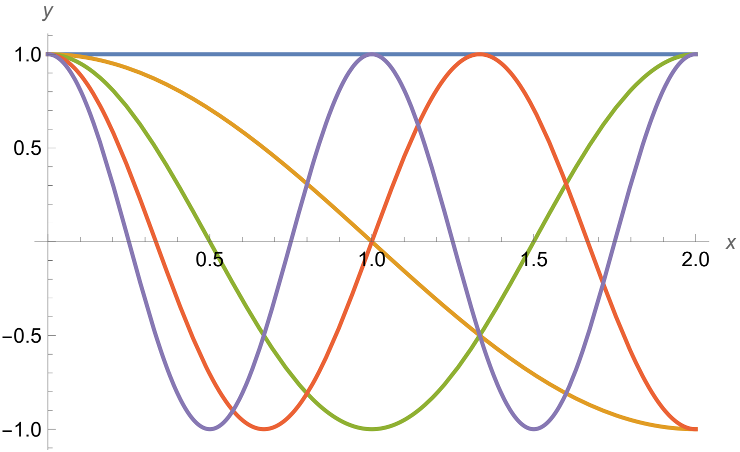
- Imposing the boundary conditions gives

$$\begin{aligned} y'(0) = 0 &\implies c_2 = 0 && \implies y(x) = c_1 \cos(k\,x) \\ y'(L) = 0 &\implies -\,k\,c_1 \sin(k\,L) = 0. && \end{aligned}$$

- For every integer $n \geq 0$, we have a solution

$$k_n = \frac{n\,\pi}{L}, \quad \lambda_n = \left(\frac{n\,\pi}{L}\right)^2, \quad y_n(x) = \cos\left(\frac{n\,\pi\,x}{L}\right).$$

Plots of first few eigenfunctions with $L = 2$



- $y_0(x)$ in blue
- $y_1(x)$ in orange
- $y_2(x)$ in green
- $y_3(x)$ in red
- $y_4(x)$ in purple

Eigenvalue problems

Theorem

Let $L > 0$ and consider the following five eigenvalue problems

$$\begin{aligned}
y'' + \lambda y &= 0, & y(0) &= 0, & y(L) &= 0; & (1)\\
y'' + \lambda y &= 0, & y'(0) &= 0, & y'(L) &= 0; & (2)\\
y'' + \lambda y &= 0, & y(0) &= 0, & y'(L) &= 0; & (3)\\
y'' + \lambda y &= 0, & y'(0) &= 0, & y(L) &= 0; & (4)\\
y'' + \lambda y &= 0, & y(-L) &= y(L), & y'(-L) &= y'(L). & (5)
\end{aligned}$$

- *None of these eigenvalue problems admit negative eigenvalues.*
- *Only eigenvalue problems (2) and (5) have $\lambda_0 = 0$ as an eigenvalue, with corresponding eigenfunction $y_0(x) = 1$.*
- *Each of the five families has a* **countable** *number of* **positive eigenvalues**,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \dots$$

with $\lambda_n \to \infty$ as $n \to \infty$.

Orthogonality

Fix an interval $[a, b] \subseteq \mathbb{R}$. We say that two real functions ϕ and ψ are **orthogonal** on $[a, b]$ when

$$\langle \phi, \psi \rangle := \int_a^b \phi(x)\psi(x)\,dx$$

$$\int_a^b \phi(x)\,\psi(x)\,dx = 0.$$

A countable collection of real functions

$$\{\phi_0, \phi_1, \phi_2, \phi_3 \ldots\},$$

is said to be an **orthogonal family** of functions on $[a, b]$ when

$$\int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \qquad \text{when } m \neq n,$$

and

$$\int_a^b \phi_n(x)^2\,dx = \int_a^b \phi_n(x)\phi_n(x)\,dx > 0,$$

for $m, n \geq 0$.

Infinite series of functions

Recall that, given a power series with radius of convergence $R > 0$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$\phi_n(x) = (x-x_0)^n$

we can compute the coefficients using the formula

$$a_n = \frac{f^{(n)}(x_0)}{n!} \qquad (n \geq 0).$$

Theorem

Let $\{\phi_0, \phi_1, \phi_2, \phi_3 \ldots\}$ be an **orthogonal family** of functions on $[a, b]$. Suppose a function $f(x)$ can be written as a series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x),$$

which we assume converges nicely$^{(*)}$ on $[a, b]$. Then we can compute the coefficients using

$$a_n = \frac{\int_a^b f(x)\phi_n(x)\, dx}{\int_a^b \phi_n(x)^2\, dx} \qquad (n \geq 0).$$

Proof of theorem

(∗) Nice convergence means: $$\sum_{n=0}^{\infty} |a_n| \int_a^b \phi_n(x)^2\, dx < \infty.$$

Proof: Take an $m \geq 0$, then

$$\int_a^b f(x)\phi_m(x)\,dx = \int_a^b \left(\sum_{n=0}^{\infty} a_n \phi_n(x) \right) \phi_m(x)\,dx$$

$$= \sum_{n=0}^{\infty} a_n \int_a^b \phi_n(x)\,\phi_m(x)\,dx$$

$$= a_m \int_a^b \phi_m(x) \cdot \phi_m(x)\,dx$$

Therefore

$$a_m = \frac{\int_a^b f(x)\,\phi_m(x)\,dx}{\int_a^b \phi_m(x)^2\,dx}.$$

Example 1

Recall that the eigenvalue problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(L) = 0,$$

has solutions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad (n \geq 1).$$

The set $\{y_1, y_2, y_3, \ldots\}$ forms an orthogonal family on $[0, L]$.

Check: For $m \neq n$ we have

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^L \cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)dx$$

$$= \frac{1}{2}\left[-\frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{L}\right) + \frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{L}\right)\right]_{x=0}^{x=L}$$

$$= 0.$$

$\int_0^L y_n(x)^2\,dx > 0$ for $n \geq 1$. $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

Example 2

Recall that the eigenvalue problem

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(L) = 0,$$

has solutions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad (n \geq 0).$$

The set $\{y_0, y_1, y_2, \ldots\}$ forms an orthogonal family on $[0, L]$.

Check: For $m \neq n$ we have

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^L \cos\left(\frac{(m-n)\pi x}{L}\right) + \cos\left(\frac{(m+n)\pi x}{L}\right)dx$$

$$= -\frac{1}{2}\left[\frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{L}\right) + \frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{L}\right)\right]_{t=0}^{t=L}$$

$$= 0.$$

$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$

Orthogonality of trigonometric functions on $[-L, L]$

Theorem

The collection of functions

$$\left\{1, \cos\left(\frac{1\pi x}{L}\right), \sin\left(\frac{1\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \cos\left(\frac{3\pi x}{L}\right),\right.$$

$$\left.\sin\left(\frac{3\pi x}{L}\right), \ldots, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right), \ldots\right\}$$

forms an **orthogonal family** on $[-L, L]$.

Proof:

See example 2 : $$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0 \qquad (m \neq n;\ m, n \geq 0)$$

See example 1 : $$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0 \qquad (m \neq n;\ m, n \geq 1)$$

$$\int_{-L}^{L} \underbrace{\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd function}}dx = 0 \qquad (m \geq 0,\ n \geq 1)$$

Fourier series

Suppose a function $f(x)$ on $[-L, L]$ can be written as a series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \tag{6}$$

where we assume the series on the right-hand side converges nicely.

- Then the coefficients can be computed using the theorem on page 14:

$$\begin{aligned}
\int_{-L}^{L} 1^2\,dx = 2L \quad &\Longrightarrow \quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \\
\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)^2 dx = L \quad &\Longrightarrow \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \\
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)^2 dx = L \quad &\Longrightarrow \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx.
\end{aligned}$$

- The series on the right-hand side of (6) is called a **Fourier series**.
- A large class of functions $f(x)$ can be written as in (6).
- More on Fourier series in Lecture 11-3.

summary

$$\boxed{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)^2 dx = \frac{1}{2}\int_{-L}^{L} 1 + \cos\left(2\frac{n\pi x}{L}\right) dx = \frac{1}{2}(2L+0) = L}$$

After today's lecture, you know

- how to solve certain **eigenvalue problems**,
- what **orthogonal families** of functions are,
- how to compute the coefficients in a series made out of functions from an orthogonal family,
- some important examples of orthogonal families.

$$\cos 2z = \cos z \cdot \cos z - \sin z \sin z$$
$$= \cos^2 z - (1 - \cos^2 z)$$
$$= 2\cos^2 z - 1$$

So $\cos^2 z = \frac{1}{2}(1 + \cos 2z)$

$$a_n = \frac{\int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx}{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)^2 dx} = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$