MATH2021 - Differential Equations
Week 11, Lecture 1

CONSULTATION: WED 1-2PM

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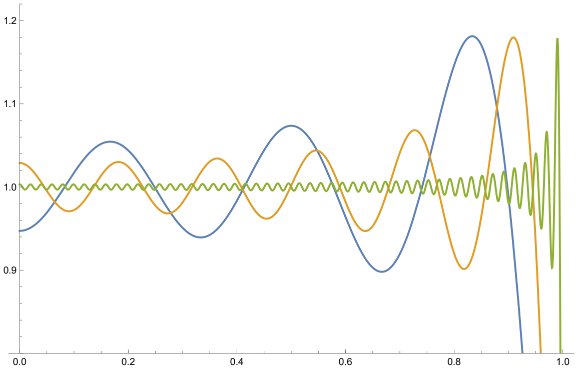

Past and present

- Last lecture we studied
  - Laplace transform
  - inverse Laplace transform
- Today:
  - Laplace transforms and t-derivatives
  - Solving initial value problems using the Laplace transform

Table (part 1)

	Function	Laplace Transform
LT 1	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$
LT 2	e^{at}	$\dfrac{1}{s-a} \qquad (s > a)$
LT 3	1	$\dfrac{1}{s} \qquad (s > 0)$
LT 4	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2} \qquad (s > 0)$
LT 5	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2} \qquad (s > 0)$
LT 6	$t^n, \quad n \geq 0$	$\dfrac{n!}{s^{n+1}} \qquad (s > 0)$
LT 7	$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$ (linearity)
LT 8	$e^{at} f(t)$	$F(s-a)$ (s-shifting)

Table (part 2) in the making

	Function	Laplace Transform
LT 9	$(-t)^n f(t)$	$\dfrac{d^n}{ds^n}F(s)$ (s-derivative)
LT 10	$f'(t)$	?
LT 11	$f''(t)$	?

- We derived the right entry of row 9 in Lecture 10-3.
- Next, we work out the right entries of rows 10 and 11.

Laplace transform of the derivative

Claim

Let $F(s) = \mathcal{L}[f](s)$ be the Laplace transform of $f(t)$, then

$$\mathcal{L}[f'](s) = s\,F(s) - f(0).$$

Let's check:

integration by parts

$\lim_{t\to\infty} f(t)e^{-st} = 0$

$$\begin{aligned}
\mathcal{L}[f'](s) &= \int_0^\infty f'(t)\,e^{-s\,t}dt \\
&= \left[f(t)\,e^{-s\,t}\right]_{t=0}^{t=\infty} - \int_0^\infty f(t)\,\frac{d}{dt}\left[e^{-s\,t}\right]dt \\
&= 0 - f(0) - \int_0^\infty f(t)\,(-s)e^{-s\,t}dt \\
&= s\int_0^\infty f(t)\,e^{-s\,t}dt - f(0) \\
&= s\,F(s) - f(0)
\end{aligned}$$

Table (part 2) in the making

	Function	**Laplace Transform**
LT 9	$(-t)^n f(t)$	$\dfrac{d^n}{ds^n} F(s)$ (s-derivative)
LT 10	$f'(t)$	$s\,F(s) - f(0)$ (t-derivative)
LT 11	$f''(t)$	?

Laplace transform of higher-order derivatives

We have shown that

$$\mathcal{L}[g'](s) = s\,\mathcal{L}[g](s) - g(0). \quad (*)$$

Therefore

APPLY (*) WITH $g = f'$

$$\mathcal{L}[f''](s) = s\,\mathcal{L}[f'](s) - f'(0),$$

APPLY (*) WITH $g = f$

$$= s\,(s\,\mathcal{L}[f](s) - f(0)) - f'(0),$$

$$= s^2\,F(s) - s\,f(0) - f'(0).$$

Similarly,

$g = f''$

$$\mathcal{L}[f'''](s) = s \cdot \mathcal{L}[f''](s) - f''(0)$$

$$= s \cdot \left(s^2\,F(s) - s\,f(0) - f'(0)\right) - f''(0)$$

$$= s^3 \cdot F(s) - s^2 f(0) - s\,f'(0) - f''(0)$$

Table (part 2)

	Function	**Laplace Transform**
LT 9	$(-t)^n f(t)$	$\dfrac{d^n}{ds^n} F(s)$ (*s*-derivative)
LT 10	$f'(t)$	$s\,F(s) - f(0)$ (*t*-derivative)
LT 11	$f''(t)$	$s^2\,F(s) - s\,f(0) - f'(0)$ (2nd *t*-derivative)

Formula for Laplace transform of *n*th derivative:

$$\mathcal{L}[f^{(n)}](s) = s^n\,F(s) - \left(s^{n-1}\,f(0) + s^{n-2}\,f'(0) + \ldots + s\,f^{(n-1)}(0) + f^{(n)}(0)\right).$$

Solving Initial value problems, example 1

Use the laplace transform to solve the following initial value problem

$$y'(t) + y(t) = e^{-t}, \qquad y(0) = 1.$$

Step 1: Apply Laplace transform to both sides of ODE.

Write $Y(s) = \mathcal{L}[y](s)$, then

$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}$

$$\begin{aligned}\frac{1}{s+1} = \mathcal{L}[e^{-t}](s) &= \mathcal{L}[y'(t) + y(t)](s) \\ &= \mathcal{L}[y'(t)](s) + \mathcal{L}[y(t)](s) \\ &= s\,Y(s) - y(0) + Y(s) \\ &= (s+1)\,Y(s) - 1.\end{aligned}$$

Linearity of $\mathcal{L}$

Step 2: Solve for $Y(s)$.

$$Y(s) = \frac{1}{s+1}\left(\frac{1}{s+1} + 1\right) = \frac{1}{(s+1)^2} + \frac{1}{s+1}.$$

Step 3: Compute $y(t) = \mathcal{L}^{-1}[Y](t)$.

s - shifting

$$
\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y](t) &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2} + \frac{1}{s+1}\right](t) \\
&= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right](t) + \underbrace{\mathcal{L}^{-1}[\frac{1}{s+1}](t)} \\
&= e^{-t}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) + e^{-t} \\
&= e^{-t}\,t + e^{-t} \\
&= (t+1)e^{-t}.
\end{aligned}
$$

Check: $y(0) = (0+1)e^{0} = 1$.

Solving Initial value problems, example 2

Use the laplace transform to solve the following initial value problem

$$y''(t) + 2\,y'(t) + 5\,y(t) = 0, \qquad y(0) = 2, \quad y'(0) = -4.$$

Let's write $Y(s) = \mathcal{L}[y](s)$.

We apply the Laplace transform to the ODE:

$$\begin{aligned}
0 = \mathcal{L}[0](s) &= \mathcal{L}[y'' + 2y' + 5y](s) \\
&= \mathcal{L}[y''](s) + 2\,\mathcal{L}[y'](s) + 5\,\mathcal{L}[y](s) \\
&= s^2 \cdot \underline{Y(s)} - s\,y(0) - y'(0) + 2\,(s \cdot \underline{Y(s)} - y(0)) + 5\,\underline{Y(s)} \\
&= (s^2 + 2s + 5)\,Y(s) - 2s + 4 - 4 \\
&= (s^2 + 2s + 5)\,Y(s) - 2s
\end{aligned}$$

Solve for $Y(s)$:

$$Y(s) = \frac{2s}{s^2 + 2s + 5}$$

Solving Initial value problems, example 2 continued

Next, we compute $y(t)$ by applying $\mathcal{L}^{-1}$ to $Y(s)$:

$$y(t) = \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\left[\frac{2s}{s^2+2s+5}\right](t)$$

$$= \mathcal{L}^{-1}\left[\frac{2s}{(s+1)^2+4}\right](t)$$

$$\boxed{2s = 2(s+1) - 2}$$

$$= \mathcal{L}^{-1}\left[\frac{2(s+1)}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}\right](t)$$

LT 8 with $a=-1$

$$= e^{-t}\mathcal{L}^{-1}\left[\frac{2s}{s^2+4} - \frac{2}{s^2+4}\right](t)$$

$$= 2e^{-t}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right](t) - e^{-t}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right](t)$$

$$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

$$= e^{-t}(2\cos 2t - \sin 2t)$$

Use the laplace transform to solve the following initial value problem

$$y'''(t) + y(t) = 0, \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1.$$

We write $Y(s) = \mathcal{L}[y](s)$, and apply the Laplace transform to the ODE:

$$\begin{aligned} 0 = \mathcal{L}[0](s) &= \mathcal{L}[y''' + y](s) \\ &= \mathcal{L}[y'''](s) + \mathcal{L}[y](s) \\ &= s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) + Y(s) \\ &= (s^3+1) Y(s) - s^2 + 1 \end{aligned}$$

Solve for $Y(s)$:

$$Y(s) = \frac{s^2 - 1}{s^3 + 1} = \frac{(s-1)(s+1)}{(s^2 - s + 1)(s+1)} = \frac{s-1}{s^2 - s + 1}$$

Solving Initial value problems, example 3 continued

Next, we compute $y(t) = \mathcal{L}^{-1}[Y](t)$:

$$y(t) = \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\left[\frac{s-1}{s^2 - s + 1}\right](t) \qquad s - 1 = s - \tfrac{1}{2} - \tfrac{1}{2}$$

$$= \mathcal{L}^{-1}\left[\frac{s-1}{(s-\frac{1}{2})^2 + \frac{3}{4}}\right](t)$$

$$= \mathcal{L}^{-1}\left[\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2 + \frac{3}{4}} - \frac{1}{2}\,\frac{1}{(s-\frac{1}{2})^2 + \frac{3}{4}}\right](t)$$

$$= e^{\frac{1}{2}t}\mathcal{L}^{-1}\left[\frac{s}{s^2 + \frac{3}{4}} - \frac{1}{2}\,\frac{1}{s^2 + \frac{3}{4}}\right](t)$$

LT 4, $\omega = \frac{\sqrt{3}}{2}$

$$= e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - e^{\frac{1}{2}t}\,\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\,\mathcal{L}^{-1}\left[\frac{\frac{\sqrt{3}}{2}}{s^2 + \frac{3}{4}}\right](t)$$

LT 5, $\omega = \frac{\sqrt{3}}{2}$

$$= e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - e^{\frac{1}{2}t}\,\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Optional fourth example

Optional fourth example continued

summary

After today's lecture, you

- know how the Laplace transform acts on derivatives,
- are able to solve initial value problems using the Laplace transform.