MATH2021 - Differential Equations
Week 10, Lecture 3

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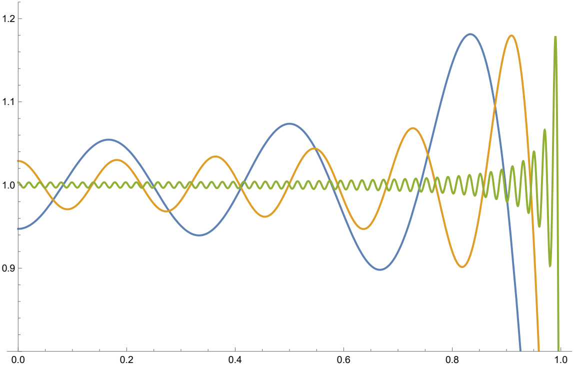


Past and present

- Last lecture was all about power series.
- Today:
  - the Laplace transform
  - the inverse Laplace transform

The Laplace transform

Let $f(t)$ be a function defined for $t \geq 0$.

The **Laplace transform** of f is defined by

$$\mathcal{L}[f](s) := \int_0^\infty f(t)e^{-st}dt,$$

for $s \in \mathbb{R}$ large enough such that the improper integral converges.

The output $\mathcal{L}[f](s)$ is a function that depends on s.

Pierre-Simon Laplace

Pierre-Simon Laplace as chancellor of the Senate under the First French Empire

Born	23 March 1749 Beaumont-en-Auge, Normandy, Kingdom of France
Died	5 March 1827 (aged 77) Paris, Kingdom of France

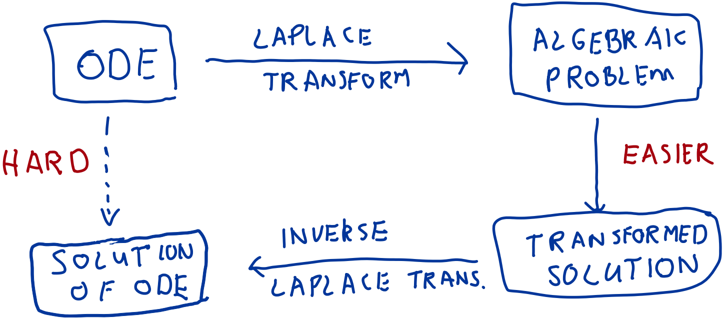
The conjugation principle

If $F(s) = \mathcal{L}[f](s)$ is the Laplace transform of $f(t)$, then, by definition,

$$f(t) = \mathcal{L}^{-1}[F](t),$$

is the **inverse Laplace transform** of $F(s)$.

The Laplace transform (and its inverse) provide a systematic way to solve ODEs with constant coefficients of **any order**.



Laplace transform of exponentials

Claim

For $a \in \mathbb{R}$ and $s > a$:

$$\mathcal{L}[e^{a\,t}](s) = \frac{1}{s-a}.$$

Let's compute

$$\begin{aligned}
\mathcal{L}[e^{a\,t}](s) &= \int_0^\infty e^{a\,t} \cdot e^{-s\,t}\,dt \\
&= \int_0^\infty e^{-(s-a)t}\,dt \\
&= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\,dt \\
&= \lim_{N\to\infty} \left[-\frac{1}{s-a} e^{-(s-a)t}\right]_{t=0}^{t=N} \\
&= \lim_{N\to\infty} -\frac{1}{s-a} e^{-(s-a)N} + \frac{1}{s-a} \\
&= 0 + \frac{1}{s-a} = \frac{1}{s-a}
\end{aligned}$$

Table in the making

	Function	Laplace Transform
LT 1	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
LT 2	e^{at}	$\frac{1}{s-a} \qquad (s > a)$
LT 3	1	?
LT 4	$\cos \omega t$	?
LT 5	$\sin \omega t$	?
LT 6	$t^n, \quad n \geq 0$	?
LT 7	$a f(t) + b g(t)$	?
LT 8	$e^{at} f(t)$	?

Laplace transform of 1

Claim

For $s > 0$:

$$\mathcal{L}[1](s) = \frac{1}{s}.$$

Let's compute

$$\begin{aligned}
\mathcal{L}[1](s) &= \int_0^\infty 1 \cdot e^{-s\,t} dt \\
&= \lim_{N\to\infty} \int_0^N e^{-s\,t} dt \\
&= \lim_{N\to\infty} \left[-\frac{1}{s} e^{-s\,t} \right]_{t=0}^{t=N} \\
&= \lim_{N\to\infty} -\frac{1}{s} e^{-s\,N} + \frac{1}{s} \\
&= 0 + \frac{1}{s} = \frac{1}{s}.
\end{aligned}$$

Table in the making

	Function	Laplace Transform
LT 1	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$
LT 2	e^{at}	$\frac{1}{s-a} \qquad (s > a)$
LT 3	1	$\frac{1}{s}$
LT 4	$\cos \omega t$	?
LT 5	$\sin \omega t$	?
LT 6	$t^n, \quad n \geq 0$	?
LT 7	$a f(t) + b g(t)$	?
LT 8	$e^{at} f(t)$	?

Linearity of the Laplace transform

Claim

Let $a, b \in \mathbb{R}$ and $F(s) = \mathcal{L}[f](s)$ and $G(s) = \mathcal{L}[g](s)$, then

$$\mathcal{L}[f+g](s) = F(s) + G(s).$$

Let's compute $\mathcal{L}[a \cdot f + b \cdot g](s) = a \cdot F(s) + b \cdot G(s)$

$$\begin{aligned}
\mathcal{L}[a \cdot f + b \cdot g](s) &= \int_0^\infty (a f(t) + b\, g(t)) e^{-st}\, dt \\
&= \int_0^\infty a \cdot f(t) e^{-st}\, dt + \int_0^\infty b \cdot g(t) e^{-st}\, dt \\
&= a \cdot \mathcal{L}[f](s) + b \cdot \mathcal{L}[g](s) \\
&= a F(s) + b G(s
\end{aligned}$$

Table in the making

	Function	Laplace Transform
LT 1	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
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LT 3	1	$\frac{1}{s}$
LT 4	$\cos \omega t$	?
LT 5	$\sin \omega t$	?
LT 6	$t^n, \quad n \geq 0$	?
LT 7	$a\, f(t) + b\, g(t)$	$a\, F(s) + b\, G(s)$ (linearity)
LT 8	$e^{at} f(t)$	?

Laplace transform of cosine

Claim

For $s > 0$:

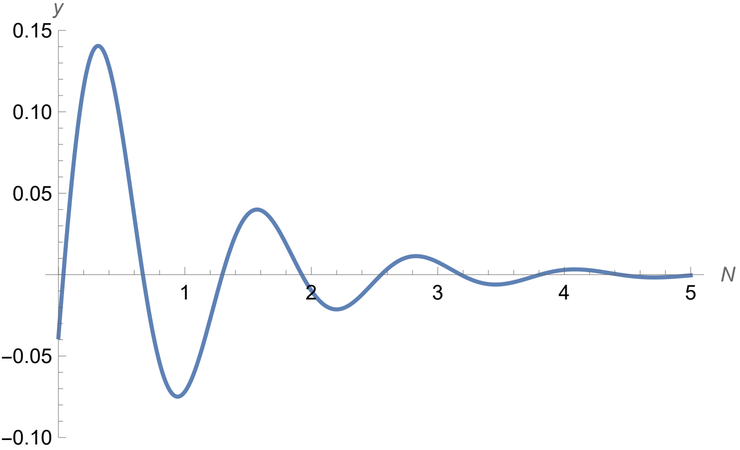
$$\mathcal{L}[\cos \omega t](s) = \frac{s}{s^2 + \omega^2}.$$

Let's compute

$$\begin{aligned}
\mathcal{L}[\cos \omega t](s) &= \int_0^\infty \cos \omega t \cdot e^{-s\,t} dt \\
&= \lim_{N\to\infty} \int_0^N \cos \omega t \cdot e^{-s\,t} dt \\
&= \lim_{N\to\infty} \left[\frac{1}{s^2+\omega^2} e^{-s\,t} \left(\omega \sin \omega t - s \cos \omega t\right) \right]_{t=0}^{t=N} \\
&= \lim_{N\to\infty} \frac{1}{s^2+\omega^2} e^{-s\,N} \left(\omega \sin \omega N - s \cos \omega N\right) - \frac{1}{s^2+\omega^2} (0 - s) \\
&= 0 + \frac{s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}.
\end{aligned}$$

integrating by parts twice

Numerical display of limit



Plot of

$$y(N) = \frac{1}{s^2 + \omega^2} e^{-sN} (\omega \sin \omega N - s \cos \omega N)$$

with $\omega = 5$ and $s = 1$.

Table in the making

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LT 3	1	$\frac{1}{s}$
LT 4	$\cos \omega t$	$\frac{s}{s^2+\omega^2} \quad (s > 0)$
LT 5	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2} \quad (s > 0)$
LT 6	$t^n, \quad n \geq 0$	?
LT 7	$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$ (linearity)
LT 8	$e^{at} f(t)$	?

Exponential shift (s-shifting)

Claim

Let $F(s) = \mathcal{L}[f](s)$, then

$$\mathcal{L}[e^{at} f](s) = F(s-a)$$

Let's compute

$$\mathcal{L}[e^{at} \cdot f](s) = \int_0^\infty e^{at} f(t) e^{-st} \, dt$$

$$= \int_0^\infty f(t) e^{at-st} \, dt$$

$$= \int_0^\infty f(t) e^{-(s-a)t} \, dt$$

$$= F(s-a) \qquad \left(F(s) = \int_0^\infty f(t) e^{-st} \, dt \right)$$

Table in the making

	Function	Laplace Transform
LT 1	$f(t)$	$F(s)=\int_0^\infty e^{-st} f(t)\,dt$
LT 2	e^{at}	$\dfrac{1}{s-a} \qquad (s>a)$
LT 3	1	$\dfrac{1}{s}$
LT 4	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2} \qquad (s>0)$
LT 5	$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2} \qquad (s>0)$
LT 6	$t^n, \quad n\geq 0$	?
LT 7	$a\,f(t)+b\,g(t)$	$a\,F(s)+b\,G(s)$ (linearity)
LT 8	$e^{at}f(t)$	$F(s-a)$ (s-shifting)

Example 1

Compute the Laplace transform of the function

$$f(t) = \cos(2t)e^{3t}.$$

We know that $\mathcal{L}[\cos(2t)](s) = \dfrac{s}{s^2+4}$

By s-shifting,

$$\mathcal{L}[\cos(2t)e^{3t}](s) = \mathcal{L}[\cos 2t](s-3)$$

$$= \frac{s-3}{(s-3)^2+4}$$

Claim

Let $F(s) = \mathcal{L}[f](s)$, then

$$\mathcal{L}[t\,f](s) = -\frac{d}{ds}F(s).$$

Let's compute

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= \int_0^\infty \frac{d}{ds}\left[f(t)\,e^{-st}\right]dt$$

$$= \int_0^\infty f(t)\,(-t)\,e^{-st}\,dt$$

$$= -\int_0^\infty t\,f(t)\,e^{-st}\,dt$$

$$= -\mathcal{L}[t\cdot f](s)$$

Higher-order s-derivatives

Higher-order s-derivatives

Let $F(s) = \mathcal{L}[f](s)$, then

$$\mathcal{L}[t^n f](s) = (-1)^n \frac{d^n}{ds^n} F(s).$$

Corollary

For $s > 0$ and $n \geq 0$:

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^n](s) = \mathcal{L}[t^n \cdot 1](s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[1](s)$$

$$= (-1)^n \frac{d^n}{ds^n} \frac{1}{s} = \frac{n!}{s^{n+1}} \quad \checkmark$$

$n=1$: $-\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$ $\qquad$ $n=2$: $\frac{d^2}{ds^2}\frac{1}{s} = \frac{d}{ds} -\frac{1}{s^2} = \frac{2}{s^3}$

Table

	Function	**Laplace Transform**
LT 1	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
LT 2	e^{at}	$\dfrac{1}{s-a} \qquad (s > a)$
LT 3	1	$\dfrac{1}{s} \qquad (s > 0)$
LT 4	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2} \qquad (s > 0)$
LT 5	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2} \qquad (s > 0)$
LT 6	$t^n, \quad n \geq 0$	$\dfrac{n!}{s^{n+1}} \qquad (s > 0)$
LT 7	$a\, f(t) + b\, g(t)$	$a\, F(s) + b\, G(s)$ (linearity)
LT 8	$e^{at} f(t)$	$F(s-a)$ (s-shifting)

Example 2

Compute

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+4}\right].$$

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s+1}{s^2+4}\right](t) &= \mathcal{L}^{-1}\left[\frac{s}{s^2+4} + \frac{1}{s^2+4}\right](t) \\
&= \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right](t) \\
&= \cos(2t) + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right](t) \\
&= \cos(2t) + \frac{1}{2}\sin 2t
\end{aligned}$$

Example 3

Compute
$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - s - 2}\right].$$

$s^2 - s - 2 = (s-2)(s+1)$

We look for A, B such that

$$\frac{1}{s^2-s-2} = \frac{A}{s-2} + \frac{B}{s+1} \implies 1 = (s+1)A + (s-2)B$$

$s = 2: \quad 1 = 3A$

$s = -1: \quad 1 = -3B$

So $A = \frac{1}{3}$, $B = -\frac{1}{3}$, therefore

Example 3 continued

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-s-2}\right](t) = \frac{1}{3}\,\mathcal{L}^{-1}\left[\frac{1}{s-2}-\frac{1}{s+1}\right](t)$$

$$= \frac{1}{3}\,\mathcal{L}^{-1}\left[\frac{1}{s-2}\right](t) - \frac{1}{3}\,\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t)$$

$$= \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$$

summary

After today's lecture, you are

- familiar with the **Laplace transform**,
- know how to compute the Laplace transform of elementary functions,
- know how to compute the inverse Laplace transform of simple rational functions.