MATH2021 - Differential Equations
Week 10, Lecture 2

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ASSIGNMENT 2
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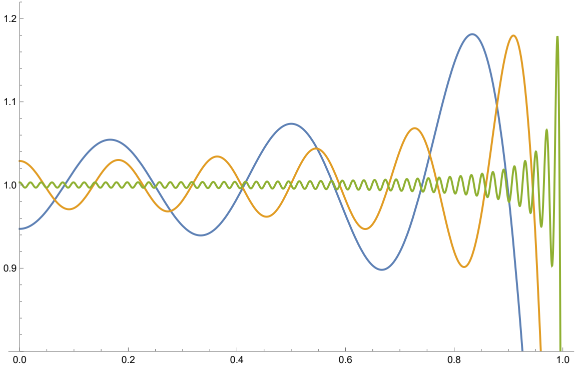


Past and present

- Last lecture:
  - method of variation of parameters
  - method of reduction of order
- Today:
  - power series
  - the power series method

Unsolvability

- We now know a number of methods to find solutions to linear 2nd order ODEs
$$y'' + p(x)y' + q(x)y = f(x). \tag{1}$$
- However, each of the methods only works under **special assumptions**, like
  - the coefficients are constant, or
  - we already have a solution to the homogeneous ODE $y'' + p(x)y' + q(x)y = 0$.
- Similar to the fact that we cannot solve many quintic equations using radicals, like
$$x^5 + 3x^4 - 2x + 7 = 0,$$
we **cannot solve** every ODE of the form (1) using integration, differentiation and our standard collection of elementary functions,
$$x^n, \quad e^x, \quad \cos x, \quad \sin x, \quad \log x, \quad \dots$$
as well as their inverses.
- This is a feature of ODEs, not a bug.

Example: Airy functions

Airy's equation

$$y'' = x\,y$$

gives rise to **new functions** $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$.

They were historically derived by Sir G.B. Airy to study supernumerary rainbows:

Sir
George Biddell Airy
KCB FRS

George Biddell Airy in 1891

Born	27 July 1801 Alnwick, Northumberland, England
Died	2 January 1892 (aged 90) Greenwich, London, England

Other methods

- We cannot solve ODEs like Airy's equation in terms of **finite** expressions involving elementary functions, integration and such.
- However, there still exist numerical and analytic methods to study them.
- One such analytical method is the **power series method**, which gives an **infinite** expression for solutions.

Power series

Power series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where x is a variable and x_0 and $a_0, a_1, a_2, \ldots$ are constants, is called a **power series** in x around x_0.

- The dummy variable n is called the **summation index**.
- The constants $a_0, a_1, a_2, \ldots$ are called **coefficients**; a_n is called the n**th coefficient** of the power series.
- Power series can be used to represent functions,

  (a) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$. (geometric series)

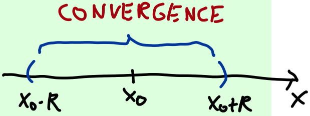
  (b) $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ for $x \in \mathbb{R}$. (exponential series)

Convergence

Radius of convergence

Any power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$



has a unique $0 \le R \le \infty$ such that

(1) the series **converges** for $|x - x_0| < R$,

(2) the series **diverges** for $|x - x_0| > R$.

R is called the **radius of convergence**.

(a) The geometric series $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$.

(b) The exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $R = \infty$.

(c) The Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} x^n$ of a function $f(x)$ can have radius of convergence $R = 0, \infty$ or $0 < R < \infty$.

Differentiating power series

Power series can be **differentiated term-wise**:

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$$P'(x) = \sum_{n=1}^{\infty} n\, a_n (x - x_0)^{n-1},$$

$$P''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

$$\vdots$$

The radius of convergence remains unchanged under differentiation.

$x_0 = 0$:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$P'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$P''(x) = 0 + 0 + 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + \dots$$

Shifting the summation index

We are free to **shift the summation index**. For example, consider

$$P'(x) = \sum_{n=1}^{\infty} n\, a_n (x - x_0)^{n-1}.$$

$n: 1, 2, 3, 4, \ldots$

$k: 0, 1, 2, 3, \ldots$

Setting $k = n - 1$, we have $n = k+1$

$$P'(x) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (x - x_0)^k.$$

Since k is just a dummy variable, we can replace it by n:

$$P'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n.$$

We shifted the summation index by 1: $n \mapsto n+1$

The power series method

To apply the **power series method** around x_0 to a 2nd order linear ODE,

$$y'' + p(x)y' + q(x)y = f(x),$$

follow the following steps:

$y'' + y = \frac{1}{x} + 1$

$x\,y'' + x \cdot y = 1 + x$

(0) If possible, rewrite the ODE in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x),$$

where $P_0(x), P_1(x), P_2(x), F(x)$ are polynomials. This step is not necessary nor always possible, but can greatly simplify the computations.

(1) Substitute a power series $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ into the ODE.

(2) Using the shifting of summation indices, ensure that all powers are aligned and read of the relations among the coefficients.

(3) recursively compute the coefficients.

Example 1

Apply the **power series method** around $x = 0$ to the ODE

$$y'' + y = 0.$$

(Note: the general solution is $y(x) = c_1 \cos x + c_2 \sin x$)

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$[n \mapsto n+2]$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substitution into the ODE gives:

$$0 = y'' + y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n$$

Example 1 continued

Therefore

$$(n+2)(n+1)\,a_{n+2} + a_n = 0 \quad \text{for all } n \geq 0.$$

We have found a <u>recurrence relation</u> for the coefficients:

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad (n \geq 0)$$

Let's compute some coefficients.

$n=0$: $\quad a_2 = -\frac{1}{2\cdot 1} a_0$

$n=1$: $\quad a_3 = -\frac{1}{3\cdot 2}\cdot a_1$

$n=2$: $\quad a_4 = -\frac{1}{4\cdot 3}\cdot a_2 = \frac{1}{4\cdot 3\cdot 2\cdot 1} a_0$

$n=3$: $\quad a_5 = -\frac{1}{5\cdot 4} a_3 = \frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1} a_1$

$n=4$: $\quad a_6 = -\frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} a_0$

Example 1 continued

$$y(x) = a_0 + a_1 \cdot x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \dots) + (a_1 x + a_3 x^3 + a_5 x^5 + \dots)$$

$$= a_0 \underbrace{\left(1 - \frac{1}{2 \cdot 1} x^2 + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} x^4 - \dots\right)}_{\cos x} + a_1 \underbrace{\left(x - \frac{1}{3 \cdot 2 \cdot 1} x^3 + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} x^5 - \dots\right)}_{\sin x}$$

$$= a_0 \cdot \cos x + a_1 \sin x$$

Example 2 (a)

Consider the linear 2nd order ODE

$$(1 + x^2)y'' + 5\,x\,y' + 3\,y = 0.$$

(a) Substitute a power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the ODE and derive a **recurrence relation** for its coefficients.

$$3\,y(x) = \sum_{n=0}^{\infty} 3\,a_n x^n = \sum_{n=0}^{\infty} 3\,a_n x^n$$

$$5 \cdot x\,y'(x) = 5 \cdot x \cdot \sum_{n=1}^{\infty} n\,a_n x^{n-1} = \sum_{n=1}^{\infty} 5\,n\,a_n x^n = \sum_{n=0}^{\infty} 5n\,a_n x^n$$

$$x^2 \cdot y''(x) = x^2 \cdot \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)\,a_n x^n = \sum_{n=0}^{\infty} n(n-1)\,a_n x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} \underset{[n \mapsto n+2]}{=} \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2} x^n$$

Example 2 (a) continued

Substitute :

$$0 = y'' + x^2 y'' + 5xy' + 3y$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + n(n-1) \underline{a_n} x^n + 5n \underline{a_n} x^n + 3 \underline{a_n} x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + \left(n(n-1) + 5n + 3 \right) \underline{a_n} \right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1)(n+3) a_n \right] x^n$$

We conclude $(n+2)(n+1) a_{n+2} + (n+1)(n+3) a_n = 0$

So we have the recurrence relation

$$a_{n+2} = -\frac{n+3}{n+2} \cdot a_n$$

Example 2 (a) continued

Example 2 (b)

(b) Given in addition the initial conditions $y(0) = 1$, $y'(0) = 0$, derive the values of the first few coefficients a_n, $0 \leq n \leq 6$.

Since $y(x) = a_0 + a_1 \cdot x + a_2 x^2 + \dots$, we have that $y(0) = a_0$, $y'(0) = a_1$.

Therefore, the initial conditions $y(0) = 1$, $y'(0) = 0 \Rightarrow a_0 = 1$, $a_1 = 0$.

Now, consider the recursion $a_{n+2} = -\frac{n+3}{n+2} a_n$.

$n = 0$: $a_2 = -\frac{3}{2} a_0 = -\frac{3}{2}$

$n = 1$: $a_3 = -\frac{4}{3} a_1 = 0$

$n = 2$: $a_4 = -\frac{5}{4} a_2 = \frac{5 \cdot 3}{4 \cdot 2} = \frac{15}{8}$

$n = 3$: $a_5 = -\frac{6}{5} a_3 = 0$

$n = 4$: $a_6 = -\frac{7}{6} \cdot a_4 = -\frac{7 \cdot 5 \cdot 3}{6 \cdot 4 \cdot 2} = -\frac{105}{48}$

summary

After today's lecture, you know how to

- substitute a **power series** into an ODE,
- derive a **recurrence relation** for the coefficients,
- compute the first few coefficients.