MATH2021 - Differential Equations
Week 10, Lecture 1

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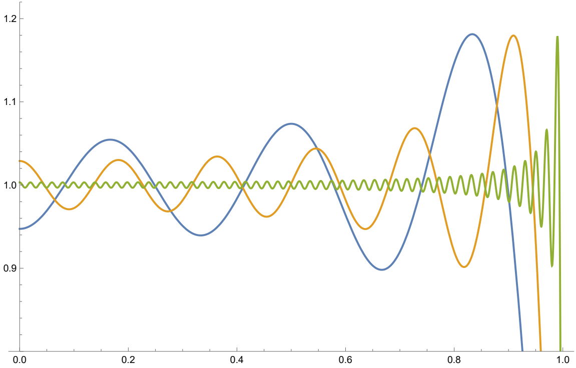

Past and present

- Last lecture was about inhomogeneous second order linear ODEs:
    - existence and uniqueness
    - general and particular solutions
    - method of undetermined coefficients
- Today we continue exploring inhomogeneous linear 2nd order ODEs:
    - method of variation of parameters
    - method of reduction of order

Limitations of the method of undetermined coefficients

The method of **undetermined coefficients** allows us to find particular solutions of ODEs of the form

$$a\,y'' + b\,y' + c\,y = f(x),$$

where $a, b, c \in \mathbb{R}$ are constants, with $a \neq 0$, only when the forcing function $f(x)$ is of the particular form

$$P(x)e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x),$$

with $\alpha, \beta, c_1, c_2 \in \mathbb{R}$ and $P(x)$ a polynomial.

In particular, it cannot be applied to the ODE

$$y'' - 3\,y' + 2\,y = \frac{e^{2x}}{1 + e^{x}}.$$

To solve this ODE, we require a different method.

The method of variation of parameters

We wish to solve the inhomogeneous linear 2nd order ODE

$$y'' + p(x)y' + q(x)y = f(x).$$

Suppose we have a fundamental set of solutions $\{y_1, y_2\}$ of the homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Lagrange's method of **variation of parameters** starts with the **ansatz**

$$y_p(x) = \color{red}{u_1(x)}y_1(x) + \color{red}{u_2(x)}y_2(x).$$

Joseph-Louis Lagrange

Born	Giuseppe Lodovico Lagrangia 25 January 1736 Turin, Kingdom of Sardinia
Died	10 April 1813 (aged 77) Paris, First French Empire

Substitution

We impose a cleverly chosen condition,

$$\boxed{u_1' y_1 + u_2' y_2 = 0,}$$

which will simplify the first and second derivative of y_p:

$$y_p = u_1 y_1 + u_2 y_2,$$

$$y_p' = u_1 y_1' + u_2 y_2' + \overbrace{u_1' y_1 + u_2' y_2}^{=0} = u_1 y_1' + u_2 y_2',$$

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'.$$

Substitution into the ODE gives

$$\begin{aligned}
f(x) =& y_p'' + p(x) y_p' + q(x) y_p \\
=& \underline{u_1 y_1''} + \underline{u_2 y_2''} + u_1' y_1' + u_2' y_2' + \underline{p(x) u_1 y_1'} + \underline{p(x) u_2 y_2'} \\
& + \underline{q(x) u_1 y_1} + \underline{q(x) u_2 y_2} \\
=& \underline{u_1}(y_1'' + p(x) y_1' + q(x) y_1) + \underline{u_2}(y_2'' + p(x) y_2' + q(x) y_2) + u_1' y_1' + u_2' y_2' \\
=& u_1 \cdot 0 + u_2 \cdot 0 + u_1' y_1' + u_2' y_2' \\
=& u_1' y_1' + u_2' y_2'.
\end{aligned}$$

$$\boxed{f(x) = u_1' y_1' + u_2' y_2'}$$

The remaining two equations

The problem has been reduced to solving the following two equations,

$$\begin{aligned} u_1' \, y_1 + u_2' \, y_2 &= 0, \\ u_1' \, y_1' + u_2' \, y_2' &= f(x). \end{aligned}$$

We can rewrite this in matrix form,

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse of the matrix on the left is

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{\mathcal{W}(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{\mathcal{W}(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ f(x) \end{bmatrix} = \frac{1}{\mathcal{W}(y_1, y_2)} \begin{bmatrix} -y_2 f(x) \\ y_1 f(x) \end{bmatrix}.$$

Formulas for the coefficients

We have derived

$$\begin{aligned} u_1'(x) &= -\frac{y_2(x)f(x)}{\mathcal{W}(y_1,y_2)(x)}, \\ u_2'(x) &= \frac{y_1(x)f(x)}{\mathcal{W}(y_1,y_2)(x)}, \end{aligned}$$

where $\mathcal{W}(y_1, y_2)$ the Wronskian of y_1 and y_2.

Integration gives

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)f(x)}{\mathcal{W}(y_1,y_2)(x)}dx, \\ u_2(x) &= \int \frac{y_1(x)f(x)}{\mathcal{W}(y_1,y_2)(x)}dx. \end{aligned}$$

Conclusion

The method of variation of parameters

Let $p(x), q(x), f(x)$ be continuous functions on an open interval $\mathrm{I} \subseteq \mathbb{R}$.

Suppose $\{y_1, y_2\}$ is a **fundamental set of solutions** of the homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0,$$

then

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

with

$$u_1(x) = -\int \frac{y_2(x)f(x)}{\mathcal{W}(y_1, y_2)(x)}dx,$$
$$u_2(x) = \int \frac{y_1(x)f(x)}{\mathcal{W}(y_1, y_2)(x)}dx,$$

is a **particular solution** to the inhomogeneous ODE

$$y'' + p(x)y' + q(x)y = f(x).$$

Example 1

Find a particular solution on $\mathrm{I} = \mathbb{R}$ to the inhomogeneous ODE

$$y'' - 3\,y' + 2\,y = \frac{e^{2x}}{1 + e^{x}}.$$

$f(x) = \frac{e^{2x}}{1+e^{x}}$

Step 1: Find a fundamental set of solutions $\{y_1, y_2\}$ of the homogeneous equation

$$y'' - 3\,y' + 2\,y = 0.$$

Characteristic polynomial: $P(\lambda) = \lambda^2 - 3\lambda + 2$

Note $P(\lambda) = (\lambda - 1)(\lambda - 2)$.

So $y_1(x) = e^{x}$, $y_2(x) = e^{2x}$ form a fundamental set of solutions.

Step 2: Compute the Wronskian of y_1 and y_2.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = e^{3x}$$

Step 3: Apply the method of variation of parameters.

We set $y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$, where

$$u_1(x) = -\int \frac{y_2 \cdot f}{W(y_1,y_2)}\,dx = -\int \frac{e^{2x}}{e^{3x}} \cdot \frac{e^{2x}}{1+e^x}\,dx = -\int \frac{e^x}{1+e^x}\,dx$$

$$= -\log(1+e^x)$$

$$u_2(x) = \int \frac{y_1 \cdot f}{W(y_1,y_2)}\,dx = \int \frac{e^x}{e^{3x}}\,\frac{e^{2x}}{1+e^x}\,dx = \int \frac{1}{1+e^x}\,dx$$

$$= \int \frac{1+e^x-e^x}{1+e^x}\,dx = \int \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\,dx = \int 1 - \frac{e^x}{1+e^x}\,dx$$

$$= x - \log(1+e^x).$$ Therefore

$$y_p = -\log(1+e^x)\cdot e^x + (x - \log(1+e^x))\cdot e^{2x}$$

is a particular solution.

Example 2

Find a particular solution on $\mathrm{I} = (0, \infty)$ to the inhomogeneous ODE

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \log x.$$

Hint: $y_1(x) = x$ *and* $y_2(x) = x \log x$ *are solutions to* $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$.

Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x \log x \\ 1 & \log x + 1 \end{vmatrix} = x(\log x + 1) - x \log x = x$$

We put

$$y_p = u_1 \cdot y_1 + u_2 \cdot y_2$$

with

Example 2 continued

$$u_1 = -\int \frac{y_2 \cdot f}{W(y_1, y_2)}\,dx = -\int \frac{x\log x \cdot \log x}{x}\,dx = -\int \log(x)^2\,dx$$

$$= -\left[\log x \cdot (x\log x - x) - \int \frac{1}{x}\cdot(x\log x - x)\right] \quad (\text{integration by parts})$$

$$= -x\log(x)^2 + x\log x + \int \log x - 1\,dx$$

$$= -x\log(x)^2 + 2x\log x - 2x$$

$$u_2 = \int \frac{y_1}{W(y_1, y_2)} f\,dx = \int \frac{x\cdot\log x}{x}\,dx = \int \log x\,dx = x\log x - x$$

Putting it all together, we find the particular solution

$$y_p = u_1 y_1 + u_2 y_2 = (-x\log(x)^2 + 2x\log x - 2x)\cdot x + (x\log x - x)\cdot x\log x$$

$$= x^2\log x - 2x^2$$

Method of reduction of order

Consider again the inhomogeneous linear 2nd order ODE

$$y'' + p(x)y' + q(x)y = f(x).$$

The method of variation of parameters requires us to know **two** linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

What if we only know **one** non-trivial solution $y_1(x)$ to the homogeneous equation?

In that case, the **method of reduction of order** allows us to transform

2nd order problem $\rightarrow$ 1st order problem,

and find the general solution.

The method in theory 1/2

Consider the **ansatz**

$$y(x) = u(x)y_1(x).$$

The first and second derivative are given by

$$y' = u'y_1 + uy_1',$$

$$y'' = u''y_1 + 2u'y_1' + uy_1''.$$

Substitution into inhomogeneous ODE gives

$$\begin{aligned}
f(x) &= y'' + p(x)y' + q(x)y, \\
&= u''y_1 + 2u'y_1' + uy_1'' + u'p(x)y_1 + up(x)y_1' + uq(x)y_1, \\
&= u''y_1 + u'(2y_1' + p(x)y_1) + u\underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{=0} \\
&= u''y_1 + u'(2y_1' + p(x)y_1) + u \cdot 0 \\
&= u''y_1 + u'(2y_1' + p(x)y_1).
\end{aligned}$$

$$f(x) = u''y_1 + u'(2y_1' + p(x)y_1)$$

Note: this is a **1st order** ODE in $v := u'$.

The method in theory 2/2

We have the following 1st order ODE for $v = u'$,

$$v' + \left(2\frac{y_1'(x)}{y_1(x)} + p(x)\right) v = \frac{f(x)}{y_1(x)}.$$

We can solve this ODE using the **method of variation of parameters** for 1st order ODEs. The general solution is

$$v(x; c_1) = y_1(x)^{-2} e^{-\int p(x)dx} \left(\int y_1(x) f(x) e^{\int p(x)dx} + c_1 \right)$$

with one arbitrary constant c_1.

Next, we integrate to obtain

$$u(x; c_1, c_2) = \int v(x; c_1) dx + c_2.$$

Putting it all together, we obtain the **general solution**

$$y(x) = \left(\int v(x; c_1) dx + c_2 \right) y_1(x)$$

to the original inhomogeneous 2nd order ODE, with two arbitrary constants c_1,c_2.

The method of reduction of order in practice

To find the general solution of

$$y'' + p(x)y' + q(x)y = f(x),$$

given that you know a solution $y_1(x)$ of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

follow the following steps.

(1) Substitute **ansatz** $y(x) = u(x)y_1(x)$ into the ODE to obtain a **1st order** ODE for $v := u'$.

(2) Find the general solution to the 1st order ODE for v.

(3) Integrate the general solution found in step (2) to find u.

(4) Substitute result for u into $y(x) = u(x)y_1(x)$ and simplify if possible. This is the **general solution**.

Example 3

Find the general solution on $\mathrm{I} = (0, \infty)$ to the inhomogeneous ODE

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x^2} + 1.$$

Hint: $y_1(x) = x$ *is a solution of* $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$.

Ansatz: $\quad y = u \cdot y_1 = u \cdot x$

Compute: $\quad y' = u'x + u$

$$y'' = u''x + u' + u' = u'' \cdot x + 2u'$$

Substitute:

$$\frac{1}{x^2} + 1 = y'' + \frac{1}{x}y' - \frac{1}{x^2}y$$

$$= u'' \cdot x + 2u' + \frac{1}{x}(u'x + u) - \frac{1}{x}u$$

$$= u''x + 3u'$$

Example 3 continued

We find the following 1st order ODE for $v = u'$:

$$v' + \frac{3}{x} v = \frac{1}{x^3} + \frac{1}{x}$$

Step 2: solve ODE for v.

The general solution to the homogeneous ODE $v' + \frac{3}{x} v = 0$ is

$$v_h(x) = c\, e^{-\int \frac{3}{x} dx} = c \cdot e^{-3 \log x} = \frac{c}{x^3} \quad (c \text{ arbitrary constant}).$$

Next, we consider the ansatz $v(x) = \frac{c(x)}{x^3}$.

Substitution gives:

$$\frac{1}{x^3} + \frac{1}{x} = v' + \frac{3}{x} v = \frac{c'(x)}{x^3} \underbrace{- 3 \frac{c(x)}{x^4}} + \underbrace{\frac{3}{x} \cdot \frac{c(x)}{x^3}} = \frac{c'(x)}{x^3}$$

Solve for $c'(x)$,

$$c'(x) = 1 + x^2 \implies c(x) = x + \frac{1}{3} x^3 + C_1$$

The general solution is $v(x) = \frac{c(x)}{x^3} = \frac{1}{x^2} + \frac{1}{3} + \frac{C_1}{x^3}$

Example 3 continued

Step 3: Compute $u(x)$

$$u(x) = \int v(x)\,dx = \int \frac{1}{3} + \frac{1}{x^2} + \frac{c_1}{x^3}\,dx + c_2$$

$$= \frac{1}{3}x - \frac{1}{x} - \frac{1}{2}\frac{c_1}{x^2} + c_2$$

Step 4: Substitute formula for u into ansatz.

$$y(x) = u(x) \cdot y_1(x)$$

$$= \left(\frac{1}{3}x - \frac{1}{x} - \frac{1}{2}\frac{c_1}{x^2} + c_2\right) \cdot x$$

$$= \frac{1}{3}x^2 - 1 - \frac{1}{2}\frac{c_1}{x} + c_2 x \qquad \left(\text{Set } \tilde{c}_1 = -\tfrac{1}{2}c_1\right)$$

$$= \frac{1}{3}x^2 - 1 + \frac{\tilde{c}_1}{x} + c_2 x$$

with arbitrary constants $\tilde{c}_1, c_2$, is the general solution.

summary

After today's lecture, you will be able to solve many more 2nd order inhomogeneous linear ODEs, using

- the method of **variation of parameters**,
- the method of **reduction of order**.